

Hadronic and leptonic corrections to the anomalous magnetic moment of the muon

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Outline

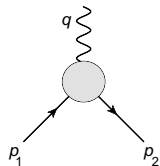
- 1 Introduction
- 2 Hadronic contribution
- 3 Leptonic correction

Anomalous magnetic moment

$$\vec{\mu} = g \frac{e}{2m} \vec{S} \quad g = 2 \cdot (1 + a)$$

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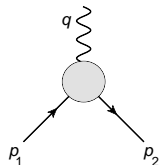


$$= -ie \bar{\psi}(p_2) \left(\gamma^\mu F_E(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2m} F_M(q^2) \right) \psi(p_1)$$

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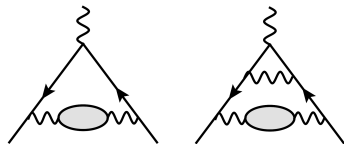
$$F_E(0) = 1, \quad F_M(0) = a$$

[PDG] $a_\mu^{\text{exp}} = 11659208.9(6.3) \cdot 10^{-10}$

$$a_\mu^{\text{th}} = 11659180.3(4.9) \cdot 10^{-10}$$

$$\Rightarrow \Delta a_\mu \approx 3 \sigma$$

Hadronic contribution

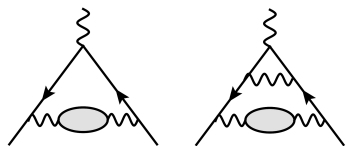


LO [Davier et al 2010] [Hagiwara et al 2011]
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NLO [Barbieri, Remiddi 1975] [Krause 1997]

NNLO

Hadronic contribution



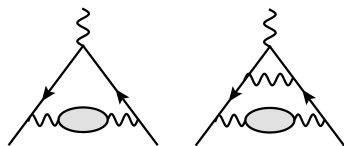
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NNLO

$$a_{\mu}^{(1)} = \int d\ell \left[P^{\omega} e^{\gamma\rho} \frac{1}{\not{p} + \not{q} + \not{\ell} - m_{\mu}} \gamma_{\omega} \frac{1}{\not{p} + \not{\ell} - m_{\mu}} e^{\gamma\lambda} \frac{g^{\rho\mu}}{\ell^2} \frac{g^{\nu\lambda}}{\ell^2} \right] (\ell^2 g_{\mu\nu} - \ell_{\mu} \ell_{\nu}) \Pi(\ell^2)$$

Hadronic contribution



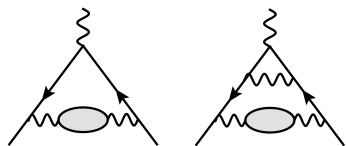
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Hadronic contribution



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$$\begin{aligned}
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 &= \int d\ell [\dots] (\ell^2 g_{\mu\nu} - \ell_\mu \ell_\nu) \cdot \frac{\ell^2}{\pi} \int_{m_\pi^2}^{\infty} \frac{1}{\ell^2 - s} \text{Im}\Pi(s) \frac{ds}{s} \\
 &= \frac{1}{3} \left(\frac{\alpha}{\pi} \right)^2 \int_{m_\pi^2}^{\infty} K^{(1)}(s) R(s) \frac{ds}{s}
 \end{aligned}$$

Hadronic calculation

$$a_{\mu}^{(3)} = \frac{1}{3} \left(\frac{\alpha}{\pi} \right)^4 \int_{m_{\pi}^2}^{\infty} ds K^{(3)}(s) R(s) / s$$



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compilation from [Nomura, Teubner]



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- resonances J/Ψ , Ψ' and $\Upsilon(nS)$
in narrow-width approximation

$$R(s) \sim \Gamma_{ee} M_R \cdot \delta(s - M_R^2)$$

Hadronic result

$$a_{\mu}^{\text{NLO}} \cdot 10^{10} = \begin{array}{ccc} \begin{array}{c} \text{Diagram 1: Triangle with photon and gluon} \\ \text{Value: } -20.90 \end{array} & \begin{array}{c} \text{Diagram 2: Triangle with photon and charm quark} \\ \text{Value: } +10.68 \end{array} & \begin{array}{c} \text{Diagram 3: Triangle with photon and bottom quark} \\ \text{Value: } +0.35 \end{array} \\ & & = -9.87(9) \\ & & -9.84(6) \end{array}$$

[Hagiwara, Liao, Martin, Nomura, Teubner 2011]

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$$a_{\mu}^{\text{NNLO}} \cdot 10^{10} = \begin{array}{ccccc} \begin{array}{c} \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \\ \text{Diagram 8} \end{array} & = & 0.80 & -0.41 & +0.91 & -0.06 & +0.0005 \\ & = & 1.24(1) & & & & \end{array}$$

[Kurz, Liu, Marquard, Steinhauser 2014]

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[Kurz, Liu, Marquard, Steinhauser 2014]

$$\Delta a_{\mu}^{\text{old}} \Rightarrow \Delta a_{\mu}^{\text{new}} = a_{\mu}^{\text{exp}} - a_{\mu}^{\text{th}} - a_{\mu}^{\text{NNLO}}$$

$$= 23.7(8.6) \cdot 10^{-10}$$

$$2.9 \sigma \Rightarrow 2.7 \sigma$$

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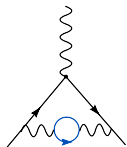
$$= 23.7(8.6) \cdot 10^{-10}$$

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$$a_{\mu}^{\text{HLbL,NLO}} = 0.3(2) \cdot 10^{-10}$$

[Colangelo, Hoferichter, Nyffeler, Passera, Stoffer 2014]

Leptonic correction



$$a_{\mu}(\tau) \sim \mathcal{O}\left(\frac{m_{\mu}^2}{m_{\tau}^2}\right)$$

$$a_{\mu}(e) \sim \mathcal{O}\left(\ln \frac{m_{\mu}}{m_e}\right)$$

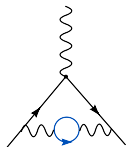
2 ℓ [Elend 1966]

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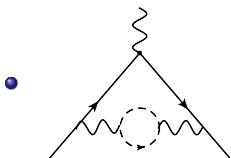
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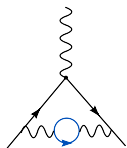
Asymptotic expansion



$$l_1^2 \approx l_2^2 \approx m_\mu^2 = q^2$$

$$(m_e = 0, 4\ell \text{ [Marquard, Smirnov, Smirnov, Steinhauser] })$$

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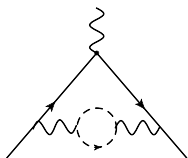
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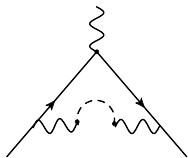
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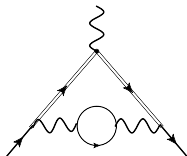
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$$l_1^2 \approx m_\mu^2 \gg l_2^2 \approx m_e^2$$

$$(l_1 l_2)^2 \rightarrow \frac{l_1^2 l_2^2}{d}$$

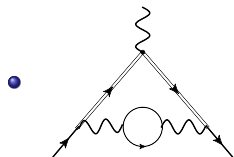
Calculation methods



$$l_1^2 \approx l_2^2 \approx m_e^2$$

$$\frac{1}{(\ell+q)^2 - m_\mu^2} = \frac{1}{\ell^2 + 2\ell q} = \frac{1}{2\ell q} \sum_n \left(\frac{-\ell^2}{2\ell q} \right)^n$$

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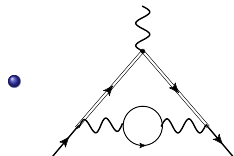
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Integration-by-Parts:

$$\int dk \frac{\partial}{\partial k_i^\nu} k_j^\nu f(k^2) = 0 \quad \Rightarrow \quad \text{arb. Integral} = \sum_n c_n \text{ MasterIntegral}_n$$

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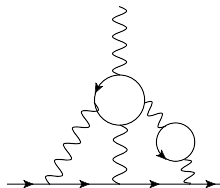
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Preliminary result:

$$121.6(2.1) n_f^2 \left(\frac{\alpha}{\pi} \right)^4 + \mathcal{O} \left(\frac{m_e}{m_\mu} \right)$$

$$\leftrightarrow 123.78551(44) \quad [\text{Aoyama, Hayakawa, Kinoshita, Nio 2012}]$$

$$[4\ell, \text{QED} : 130.8796(63)]$$



Heavy-lepton result

$$a_{\mu}(\tau) = 0.0078 \cdot 10^{-2} \cdot \left(\frac{\alpha}{\pi}\right)^2 + 0.0361 \cdot 10^{-2} \cdot \left(\frac{\alpha}{\pi}\right)^3 + A_{\mu}^{(8)}(\tau) \cdot \left(\frac{\alpha}{\pi}\right)^4$$

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$$A_\mu^{(8)}(\tau) = \left(\frac{m_\mu}{m_\tau}\right)^2 \left(\frac{4\pi^2\zeta_3}{15} - \frac{52\ln^5(2)}{675} - \frac{3851\pi^2}{3600} + \dots \right. \\ \left. + \ln \frac{m_\mu^2}{m_\tau^2} \left(-\frac{38891}{12150} + \frac{19\pi^2}{135} + \frac{3\zeta_3}{2} \right) + \frac{359}{1080} \ln^2 \frac{m_\mu^2}{m_\tau^2} \right) \\ + \left(\frac{m_\mu}{m_\tau}\right)^3 \frac{\pi^2}{90} + \dots + \mathcal{O}\left(\left(\frac{m_\mu}{m_\tau}\right)^8\right)$$

$$\approx 4.24941(2)(53) \cdot 10^{-2} \quad \leftrightarrow \quad 4.234(12) \cdot 10^{-2}$$

[Kurz, Liu, Marquard, Steinhauser 2014]

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$$a_\mu(\tau) = (4.213 + 0.045 + 0.012) \cdot 10^{-10} \quad \leftrightarrow \quad a_{\text{uni}}^{5\ell} = 0.006 \cdot 10^{-10}$$

Conclusions

- New result for a_μ^{had} at NNLO
⇒ significant change of Δa_μ by 0.2σ
- Work on $a_\mu(e)$ at $\mathcal{O}(\alpha^4)$ in progress,
first preliminary result of n_f^2 -term in agreement with literature

BACKUP

diagram group	$10^2 \cdot A_{\mu}^{(8)}(\tau)$	
	our work	ref.
I(a)	0.00324281(2)	0.0032(0)
I(b) + I(c) + II(b) + II(c)	-0.6292808(6)	-0.6293(1)
I(d)	0.0367796(4)	0.0368(0)
III	4.5208986(6)	4.504(14)
II(a) + IV(d)	-2.316756(5)	-2.3197(37)
IV(a)	3.851967(3)	3.8513(11)
IV(b)*	0.612661(5)	0.6106(31)
IV(c)	-1.83010(1)	-1.823(11)

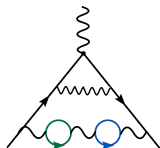
* analytical check of order $\left(\frac{m_{\mu}}{m_{\tau}}\right)^2$ [Boughezal, Melnikov 2011] [Kataev 2012]

$g - 2$ of electron

$$a_e^{\text{QED}} = a_{\text{uni}} + a_e(\mu) + a_e(\tau) + a_e(\mu, \tau)$$

$$\begin{aligned} a_e^{(8)}(\mu) &= 9.161970703(2)(372) \cdot 10^{-4} \cdot \left(\frac{\alpha}{\pi}\right)^4 \leftrightarrow 9.222(66) \cdot 10^{-4} \cdot \left(\frac{\alpha}{\pi}\right)^4 \\ &= 2.7 \cdot 10^{-14} \approx \delta a_e^{\text{exp}}/10 \end{aligned} \quad [\text{Aoyama, Hayakawa, Kinoshita, Nio 2012}]$$

$$A_e^{(8)}(\tau) = 7.42924(0)(118) \cdot 10^{-6} \leftrightarrow 7.38(12) \cdot 10^{-6}$$



$$A_e^{(8)}(\mu, \tau) = \frac{m_e^2}{m_\tau^2} \left(\frac{89}{810} \ln^2 \frac{m_\mu^2}{m_\tau^2} + \dots \right) + \mathcal{O}\left(\frac{1}{M^6}\right)$$

$$\approx 7.4687(26)(10) \cdot 10^{-7} \leftrightarrow 7.465(18) \cdot 10^{-7}$$

[Aoyama, Hayakawa, Kinoshita, Nio 2012]



$$a_e^{\text{had, NNLO}} = 2.8(1) \cdot 10^{-14}$$