Hadronic and leptonic corrections to the anomalous magnetic moment of the muon

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Outline

Anomalous magnetic moment

$\vec{\mu} = g \, \frac{e}{2m} \, \vec{s} \qquad g = 2 \cdot (1 + a)$

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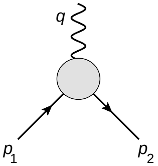

$$= \; -ie \; \bar{\psi}(p_2) \; \left(\gamma^\mu F_E(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2m} F_M(q^2) \right) \; \psi(p_1)$$

$$F_E(0) = 1, \qquad F_M(0) = a$$

Anomalous magnetic moment

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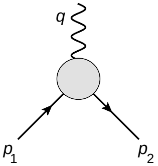

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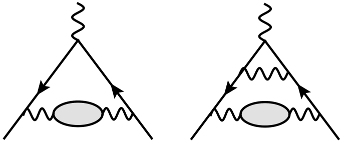
[PDG] $a_\mu^{\text{exp}} = 11659208.9(6.3) \cdot 10^{-10}$

$a_\mu^{\text{th}} = 11659180.3(4.9) \cdot 10^{-10}$

$\Rightarrow \ \Delta a_\mu \ \approx \ 3 \ \sigma$

Hadronic contribution



LO [Davier et al 2010] [Hagiwara et al 2011] [Jegerlehner et al 2011] [Benayoun et al 2012]

NLO [Barbieri, Remiddi 1975] [Krause 1997]

NNLO

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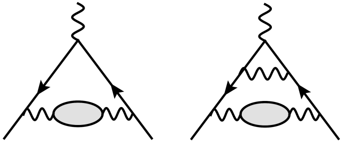

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$$a_\mu^{(1)} = \int d\ell \left[P^\omega e\gamma_\rho \frac{1}{\not{p}+\not{q}+\not{\ell}-m_\mu} \gamma_\omega \frac{1}{\not{p}+\not{\ell}-m_\mu} e\gamma_\lambda \frac{g^{\rho\mu}}{\ell^2} \frac{g^{\nu\lambda}}{\ell^2} \right] \left(\ell^2 g_{\mu\nu} - \ell_\mu \ell_\nu\right) \Pi(\ell^2)$$

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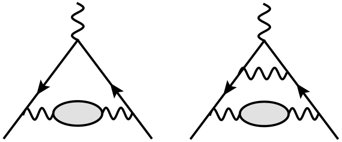

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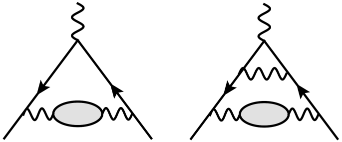
$$= \int d\ell \, [\ldots] \left(\ell^2 g_{\mu\nu} - \ell_\mu \ell_\nu\right) \cdot \frac{\ell^2}{\pi} \int_{m_\pi^2}^\infty \frac{1}{\ell^2 - s} \, \mathrm{Im}\Pi(s) \, \frac{ds}{s}$$

Hadronic contribution



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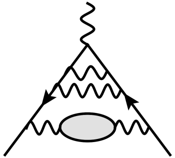
$$= \frac{1}{3} \left(\frac{\alpha}{\pi}\right)^2 \int_{m_\pi^2}^\infty K^{(1)}(s) \, R(s) \, \frac{ds}{s}$$

Hadronic calculation

$$a_\mu^{(3)} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^4 \int_{m_\pi^2}^{\infty} ds\ K^{(3)}(s)\ R(s)\ /\ s$$



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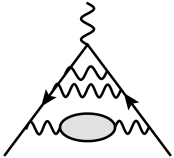

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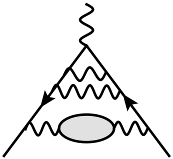

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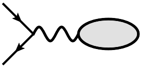

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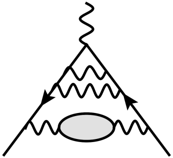

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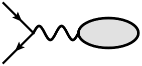
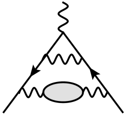
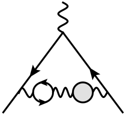
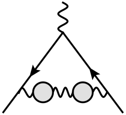


- resonances $J/\Psi, \Psi'$ and $\Upsilon(nS)$ in narrow-width approximation

$$R(s) \sim \Gamma_{ee}\ M_R \cdot \delta(s - M_R^2)$$

Hadronic result

$$a_\mu^{\mathrm{NLO}} \cdot 10^{10} \quad = \quad -20.90 \quad +10.68 \quad +0.35 \quad = \quad -9.87(9)$$

[Hagiwara, Liao, Martin, Nomura, Teubner 2011] $-9.84(6)$

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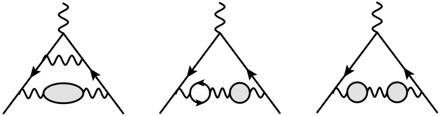

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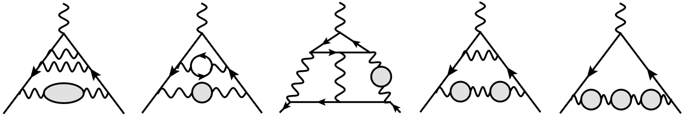

$$a_\mu^{\mathrm{NNLO}} \cdot 10^{10} = \quad 0.80 \quad -0.41 \quad +0.91 \quad -0.06 \quad +0.0005$$

$$= \quad 1.24(1)$$

[Kurz, Liu, Marquard, Steinhauser 2014]

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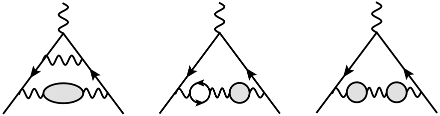

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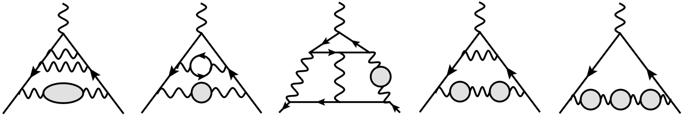

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$$\Delta a_\mu^{\rm old} \Rightarrow \Delta a_\mu^{\rm new} = a_\mu^{\rm exp} - a_\mu^{\rm th} - a_\mu^{\rm NNLO}$$

$$= 23.7(8.6) \cdot 10^{-10}$$

$2.9\ \sigma \Rightarrow\ 2.7\ \sigma$

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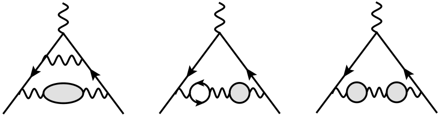

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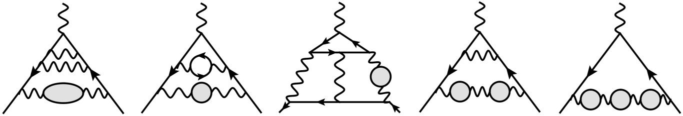

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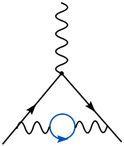
$$2.9\ \sigma \Rightarrow 2.7\ \sigma$$

$$a_\mu^{\rm HLbL,NLO} = 0.3(2) \cdot 10^{-10}$$

[Colangelo, Hoferichter, Nyffeler, Passera, Stoffer 2014]

Leptonic correction



$$a_\mu(\tau) \sim \mathcal{O}\left(\frac{m_\mu^2}{m_\tau^2}\right)$$

$$a_\mu(e) \sim \mathcal{O}\left(\ln \frac{m_\mu}{m_e}\right)$$

2ℓ [Elend 1966]

3ℓ [Laporta, Remiddi 1993; Laporta 1993]

4ℓ [Kinoshita, Nio 2003] [Lee et al 2013]

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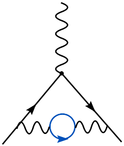

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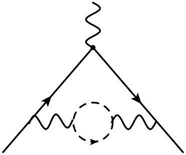
Asymptotic expansion

- 

$$\ell_1^2 \approx \ell_2^2 \approx m_\mu^2 = q^2$$

$(m_e = 0,\ \ 4\ell$ [Marquard, Smirnov, Smirnov, Steinhauser] $)$

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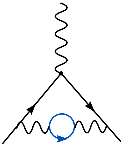

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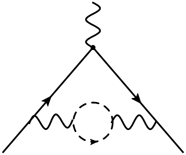
- 

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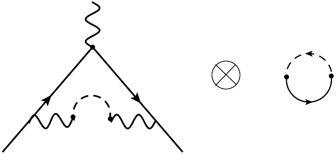
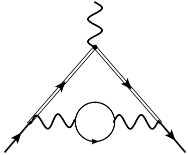
- 

  $\ell_1^2 \approx m_\mu^2 \gg \ell_2^2 \approx m_e^2$

  $(\ell_1 \ell_2)^2 \rightarrow \frac{\ell_1^2 \ell_2^2}{d}$

Calculation methods

- 

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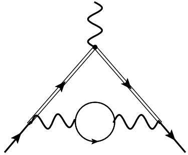
- 

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Integration-by-Parts:

$$\int dk \; \frac{\partial}{\partial k_{\rm i}^\nu} k_{\rm j}^\nu f(k^2) = 0 \quad \Rightarrow \quad \text{arb.Integral} = \sum_n c_n \, \text{MasterIntegral}_n$$

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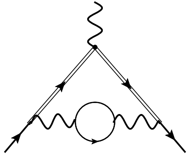
- 

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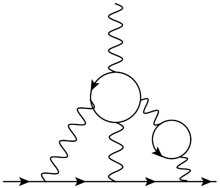
Preliminary result:

$$121.6(2.1)\ n_f^2 \left(\frac{\alpha}{\pi}\right)^4 + \mathcal{O}\left(\frac{m_e}{m_\mu}\right)$$

$\leftrightarrow$ 123.78551(44) [Aoyama, Hayakawa, Kinoshita, Nio 2012]

$[4\ell, \text{QED}:\ 130.8796(63)]$



Heavy-lepton result

$$a_\mu(\tau) = 0.0078 \cdot 10^{-2} \cdot \left(\frac{\alpha}{\pi}\right)^2 + 0.0361 \cdot 10^{-2} \cdot \left(\frac{\alpha}{\pi}\right)^3 + A_\mu^{(8)}(\tau) \cdot \left(\frac{\alpha}{\pi}\right)^4$$

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$$A_\mu^{(8)}(\tau) = \left(\frac{m_\mu}{m_\tau}\right)^2 \left(\frac{4\pi^2\zeta_3}{15} - \frac{52\ln^5(2)}{675} - \frac{3851\pi^2}{3600} + \dots \right.$$
$$\left. + \ln\frac{m_\mu^2}{m_\tau^2}\left(-\frac{38891}{12150} + \frac{19\pi^2}{135} + \frac{3\zeta_3}{2}\right) + \frac{359}{1080}\ln^2\frac{m_\mu^2}{m_\tau^2}\right)$$
$$+ \left(\frac{m_\mu}{m_\tau}\right)^3 \frac{\pi^2}{90} + \dots + \mathcal{O}\left(\left(\frac{m_\mu}{m_\tau}\right)^8\right)$$

$$\approx 4.24941(2)(53) \cdot 10^{-2} \qquad \leftrightarrow \qquad 4.234(12) \cdot 10^{-2}$$

[Kurz, Liu, Marquard, Steinhauser 2014] [Aoyama, Hayakawa, Kinoshita, Nio 2012]

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$$a_\mu(\tau) = (\underset{2\ell}{4.213} + \underset{3\ell}{0.045} + \underset{4\ell}{0.012}) \cdot 10^{-10} \qquad \leftrightarrow \qquad a_{\text{uni}}^{5\ell} = 0.006 \cdot 10^{-10}$$

Conclusions

- New result for a_μ^{had} at NNLO
  $\Rightarrow$ significant change of Δa_μ by 0.2 σ
- Work on $a_\mu(e)$ at $\mathcal{O}\left(\alpha^4\right)$ in progress,
  first preliminary result of n_f^2-term in agreement with literature

BACKUP

diagram group	$10^2 \cdot A_\mu^{(8)}(\tau)$	
	our work	ref.
I(a)	0.00324281(2)	0.0032(0)
I(b) + I(c) + II(b) + II(c)	-0.6292808(6)	-0.6293(1)
I(d)	0.0367796(4)	0.0368(0)
III	4.5208986(6)	4.504(14)
II(a) + IV(d)	-2.316756(5)	-2.3197(37)
IV(a)	3.851967(3)	3.8513(11)
IV(b)*	0.612661(5)	0.6106(31)
IV(c)	-1.83010(1)	-1.823(11)

* analytical check of order $\left(\frac{m_\mu}{m_\tau}\right)^2$ [Boughezal, Melnikov 2011] [Kataev 2012]

$g-2$ of electron

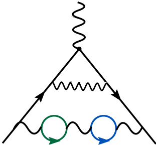
$$a_e^{\rm QED} = a_{\rm uni} + a_e(\mu) + a_e(\tau) + a_e(\mu,\tau)$$

$$a_e^{(8)}(\mu) = 9.161970703(2)(372)\cdot 10^{-4}\cdot\left(\frac{\alpha}{\pi}\right)^4 \leftrightarrow \quad 9.222(66)\cdot 10^{-4}\cdot\left(\frac{\alpha}{\pi}\right)^4$$

$$= 2.7\cdot 10^{-14} \;\approx\; \delta a_e^{\rm exp}/10$$

[Aoyama, Hayakawa, Kinoshita, Nio 2012]

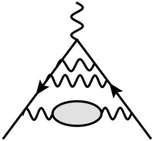
$$A_e^{(8)}(\tau) = 7.42924(0)(118)\cdot 10^{-6} \;\leftrightarrow\; 7.38(12)\cdot 10^{-6}$$



$$A_e^{(8)}(\mu,\tau) = \frac{m_e^2}{m_\tau^2}\left(\frac{89}{810}\ln^2\frac{m_\mu^2}{m_\tau^2} + \ldots\right) + \mathcal{O}\left(\frac{1}{M^6}\right)$$

$$\approx 7.4687(26)(10)\cdot 10^{-7} \leftrightarrow 7.465(18)\cdot 10^{-7}$$

[Aoyama, Hayakawa, Kinoshita, Nio 2012]



$$a_e^{\rm had,NNLO} = 2.8(1)\cdot 10^{-14}$$