Gravitational Waves

Theory, Sources and Detection

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Suggested Reading

- **Books**
  - *Gravitational Waves: Volume 1: Theory and Experiments* by Michele Maggiore Oxford University Press (2007)
  - *Gravitation and Spacetime* by Ohanian, Hans C. and Ruffini, Remo Cambridge University Preess (Sep 20, 2013)
  - *Gravitation* by Charles W. Misner, Kip S. Thorne and John Archibald Wheeler (Sep 15, 1973) W.H. Freeman
- **Review articles**
  - *Gravitational wave astronomy* F.F. Schutz, Class. Quantum Grav. 16 (1999) A131– A156
  - *Gravitational wave astronomy: in anticipation of first sources to be detected* L P Grishchuk, V M Lipunov, K A Postnov, M E Prokhorov, B S Sathyaprakash, Physics- Uspekhi 44 (1) 1 -51 (2001)
  - *The basics of gravitational wave theory* E.E. Flanagan and S.A. Hughes, New Journal of Physics 7 (2005) 204
  - *Gravitational Wave Astronomy* K.D. Kokkotas Reviews in Modern Astrophysics, Vol 20, "Cosmic Matter", WILEY-VCH, Ed.S. Roeser (2008) arXiv:0809.1602 [astro-ph]

Tensors I

Tensor Transformations

$$\tilde{b}_\mu = \sum_\nu \frac{\partial x^\nu}{\partial \tilde{x}^\mu} b_\nu \quad \text{and} \quad \tilde{a}^\mu = \sum_\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} a^\nu \tag{1}$$

$$\tilde{T}^{\alpha\beta} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu} T^{\mu\nu}\,,\; \tilde{T}^\alpha_\beta = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial \tilde{x}^\beta} T^\mu{}_\nu \;\&\; \tilde{T}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\beta} T_{\mu\nu}$$

Covariant Derivative

$$\phi_{;\lambda} = \phi_{,\lambda} \tag{2}$$

$$A_{\lambda;\mu} = A_{\lambda,\mu} - \Gamma^\rho_{\mu\lambda} A_\rho \tag{3}$$

$$T^{\lambda\mu}{}_{;\nu} = T^{\lambda\mu}{}_{,\nu} + \Gamma^\lambda_{\alpha\nu} T^{\alpha\mu} + \Gamma^\mu_{\alpha\nu} T^{\lambda\alpha} \tag{4}$$

Parallel Transport

$$\delta a_\mu = \Gamma^\lambda{}_{\mu\nu} a_\lambda dx^\nu \quad \text{for covariant vectors} \tag{5}$$

$$\delta a^\mu = -\Gamma^\mu{}_{\lambda\nu} a^\lambda dx^\nu \quad \text{for contravariant vectors} \tag{6}$$

Tensors II

The distance ds of two points $P(x^\mu)$ and $P'(x^\mu + dx^\mu)$ is given by

$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 \tag{7}$$

in two different coordinate frames ,

$$ds^2 = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = g_{\alpha\beta} dx^\alpha dx^\beta \, . \tag{8}$$

- Metric element for Minkowski spacetime

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{9}$$

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{10}$$

- For a sphere with radius R :

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{11}$$

- The metric element of a torus with radii a and b

$$ds^2 = a^2 d\phi^2 + (b + a\sin\phi)^2 d\theta^2 \tag{12}$$

- The Schwarzschild metric :

$$ds^2 = \left(1 - \frac{2}{c^2}\frac{GM}{r}\right) c^2 dt^2 - \left(1 - \frac{2}{c^2}\frac{GM}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Curvature Tensor or Riemann Tensor

$$\boxed{R^{\lambda}{}_{\beta\nu\sigma} = -\Gamma^{\lambda}{}_{\beta\nu,\sigma} + \Gamma^{\lambda}{}_{\beta\sigma,\nu} - \Gamma^{\mu}{}_{\beta\nu}\Gamma^{\lambda}{}_{\mu\sigma} + \Gamma^{\mu}{}_{\beta\sigma}\Gamma^{\lambda}{}_{\mu\nu}} \tag{13}$$

Christoffel Symbols

$$\boxed{\Gamma^{\alpha}{}_{\mu\rho} = \frac{1}{2} g^{\alpha\nu}\left(g_{\mu\nu,\rho} + g_{\nu\rho,\mu} - g_{\rho\mu,\nu}\right)} \tag{14}$$

Geodesic Equations

$$\frac{du^{\rho}}{ds} + \Gamma^{\rho}_{\mu\nu} u^{\mu} u^{\nu} = 0 \quad \text{or} \quad \frac{d^2 x^{\rho}}{ds^2} + \Gamma^{\rho}_{\mu\nu} \frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = 0$$

The Ricci and Einstein Tensors

The contraction of the Riemann tensor leads to **Ricci Tensor**

$$\begin{aligned} R_{\alpha\beta} &= R^{\lambda}{}_{\alpha\lambda\beta} = g^{\lambda\mu} R_{\lambda\alpha\mu\beta} \\ &= \Gamma^{\mu}_{\alpha\beta,\mu} - \Gamma^{\mu}_{\alpha\mu,\beta} + \Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\nu\mu} - \Gamma^{\mu}_{\alpha\nu}\Gamma^{\nu}_{\beta\mu} \end{aligned} \tag{15}$$

which is symmetric $R_{\alpha\beta} = R_{\beta\alpha}$. Further contraction leads to the **Ricci or Curvature Scalar**

$$R = R^{\alpha}{}_{\alpha} = g^{\alpha\beta} R_{\alpha\beta} = g^{\alpha\beta} g^{\mu\nu} R_{\mu\alpha\nu\beta} \, . \tag{16}$$

The following combination of Riemann and Ricci tensors is called **Einstein Tensor**

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R} \tag{17}$$

with the very important property:

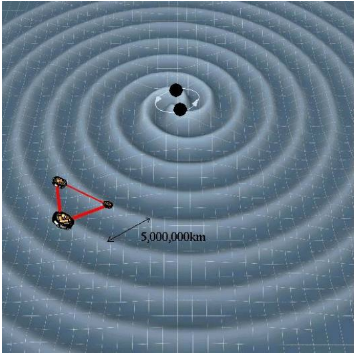
$$\boxed{G^{\mu}{}_{\nu;\mu} = \left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R \right)_{;\mu} = 0 \, .} \tag{18}$$

This results from the **Bianchi Identity** (how?)

$$\boxed{R^{\lambda}{}_{\mu[\nu\rho;\sigma]} = 0} \tag{19}$$

Flat & Empty Spacetimes

- When $R_{\alpha\beta\mu\nu} = 0$ the spacetime is flat
- When $R_{\mu\nu} = 0$ the spacetime is empty



Prove that :

$$a^{\lambda}{}_{;\mu;\nu} - a^{\lambda}{}_{;\nu;\mu} = -R^{\lambda}{}_{\kappa\mu\nu} a^{\kappa}$$

Einstein's equations

Thus the typical form of Einstein's equations is:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu} \,. \tag{20}$$

where $\Lambda = \frac{8\pi G}{c^2}\rho_v$ is the so called cosmological constant.
They can also be written as:

$$R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{21}$$

GW: Linear Theory I

Weak gravitational fields can be represented by a slightly deformed Minkowski spacetime :

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu} + O(h_{\mu\nu})^2 , \quad |h_{\mu\nu}| \ll 1 \tag{22}$$

here $h_{\mu\nu}$ is a small metric perturbation.
The indices will be raised and lowered by $\eta_{\mu\nu}$ i.e.

$$h^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu} h_{\mu\nu} \tag{23}$$

$$h = \eta^{\mu\nu} h_{\mu\nu} \tag{24}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{25}$$

and we will define the traceless ($\phi_{\mu\nu}$) tensor:

$$\phi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h . \tag{26}$$

The Christoffel symbols & the Ricci tensor will become :

$$\Gamma^{\lambda}_{\mu\nu} \approx \frac{1}{2}\eta^{\lambda\rho}\left(h_{\rho\nu,\mu} + h_{\mu\rho,\nu} - h_{\mu\nu,\rho}\right) \tag{27}$$

$$R_{\mu\nu} \approx \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} \approx \frac{1}{2}\left(h^{\alpha}{}_{\nu,\mu\alpha} + h^{\alpha}{}_{\mu,\nu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}\right) \tag{28}$$

$$R = \eta^{\mu\nu} R_{\mu\nu} \approx h_{\alpha\beta}{}^{\alpha\beta} - h^{\alpha,\beta}_{\alpha,\beta} \tag{29}$$

Finally, Einstein tensor gets the form:

$$G^{(1)}_{\mu\nu} = \frac{1}{2}\left(h^{\alpha}{}_{\nu,\mu\alpha} + h^{\alpha}{}_{\mu,\nu\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h^{\alpha}{}_{\alpha,\mu\nu}\right) - \eta_{\mu\nu}\left(h_{\alpha\beta}{}^{,\alpha\beta} - h^{\alpha}_{\alpha}{}^{,\beta}_{,\beta}\right) \quad (30)$$

Einstein's equations reduce to (how?):

$$-\phi_{\mu\nu}{}^{,\alpha}_{,\alpha} - \eta_{\mu\nu}\phi_{\alpha\beta}{}^{,\alpha\beta} + \phi_{\mu\alpha}{}^{,\alpha}_{,\nu} + \phi_{\nu\alpha}{}^{,\alpha}_{,\mu} = \kappa T_{\mu\nu} \quad (31)$$

Then by using the so called Hilbert (or Harmonic or De Donder) gauge similar to Lorenz gauge ($A^{\alpha}{}_{,\alpha} = A_{\alpha}{}^{,\alpha} = 0$) in EM [1]

$$\phi^{\mu\alpha}{}_{,\alpha} = \phi_{\mu\alpha}{}^{,\alpha} = 0 \quad (32)$$

we come to the following equation:

$$\phi_{\mu\nu}{}^{,\alpha}_{,\alpha} \equiv \Box\phi_{\mu\nu} \equiv -\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi_{\mu\nu} = -\kappa T_{\mu\nu} \quad (33)$$

which is a simple wave equations describing ripples of spacetime propagating with the speed of light (why?). These ripples are called gravitational waves.

[1] The De Donder gauge is defined in a curved background by the condition $\partial_\mu(g^{\mu\nu}\sqrt{-g}) = 0$

GW: about Gauge conditions (*)

By careful choice of coordinates the linearized Einstein equations can be simplified. We can fix $\eta_{\mu\nu} = diag(-1,1,1,1)$ and make small changes in the coordinates that leave $\eta_{\mu\nu}$ unchanged but induce small changes in $h_{\mu\nu}$.
For example lets consider a change of the form:

$$x'^{\mu} = x^{\mu} + \xi^{\mu} \tag{34}$$

where ξ^{μ} are 4 small arbitrary functions of the same order as $h^{\mu\nu}$. Then

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \delta^{\mu}{}_{\nu} + \partial_{\nu}\xi^{\mu} \quad \text{and} \quad \frac{\partial x^{\mu}}{\partial x'^{\nu}} = \delta^{\mu}{}_{\nu} - \partial_{\nu}\xi^{\mu}$$

Thus, the metric transforms as:

$$\begin{aligned} g'_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma} &= \left(\delta^{\rho}_{\mu} - \partial_{\mu}\xi^{\rho}\right)\left(\delta^{\sigma}_{\nu} - \partial_{\nu}\xi^{\sigma}\right)\left(\eta_{\rho\sigma} + h_{\rho\sigma}\right) \\ &\approx \eta_{\mu\nu} + h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} = \eta_{\mu\nu} + h'_{\mu\nu} \end{aligned} \tag{35}$$

Then in the new coordinate system we get

$$h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \tag{36}$$

This transformation is called gauge transformation.

GW: about Gauge conditions II (*)

This is analogous to the gauge transformation in Electromagnetism.
If A_μ is a solution of the EM field equations then another solution **that describes precisely the same physical situation** is given by

$$A_\mu^{(\mathrm{new})} = A_\mu - \psi_{,\mu} \tag{37}$$

where ψ is any scalar field.
Then the gauge condition $A^{(\mathrm{new})\mu}{}_{,\mu} = A^{(\mathrm{new})}{}_\mu{}^{,\mu} = 0$ means that $\psi^{,\mu}{}_{,\mu} = \psi_{,\mu}{}^{,\mu} = \Box\psi = A^\mu{}_{,\mu}$.

From (36) it is clear that if $h_{\mu\nu}$ is a solution to the linearised field equations then the same physical situation is also described by

$$\phi_{\mu\nu}^{(\mathrm{new})} = \phi_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} = \phi_{\mu\nu} - \Xi_{\mu\nu} \tag{38}$$

NOTE

- This is a gauge transformation and not a coordinate one
- We are still working on the same set of coordinates x^μ and have defined a new tensor $\phi_{\mu\nu}^{(\mathrm{new})}$ whose components in this basis are given by (38).

GW: about Gauge conditions III (*)

We can easily see that from (36) or (38) we can get

$$\phi^{(\text{new})\,\mu\rho}{}_{,\rho} = \phi^{\mu\rho}{}_{,\rho} - \Box\xi^{\mu} \tag{39}$$

Therefore, if we choose the function ξ^{μ} so that to satisfy

$$\Box\xi^{\mu} = \phi^{\mu\rho}{}_{,\rho} \tag{40}$$

we get the Hilbert gauge

$$\phi^{(\text{new})\,\mu\rho}{}_{,\rho} = 0 \tag{41}$$

NOTE: This gauge condition is preserved by any further gauge transformation of the form (38) provided that the functions ξ^{μ} satisfy $\Box\xi^{\mu} = 0$ or equivalently $\Box\Xi^{\mu\nu} = 0$.

- The choice of the Hilbert gauge $\phi^{\mu\nu}{}_{,\nu} = 0$, gives **4** conditions that reduces the **10** independent components of the symmetric tensor $h_{\mu\nu}$ to **6**!
- Eqn (38) tells us that, from the **6** independent components of $\phi_{\mu\nu}$ which satisfy $\Box\phi_{\mu\nu} = 0$, we can subtract the functions $\Xi_{\mu\nu}$, which depend on 4 independent arbitrary functions ξ_μ satisfying the same equation $\Box\Xi_{\mu\nu} = 0$.
- This means that we can choose the functions ξ_μ so that as to impose **4** conditions on $\phi_{\mu\nu}$.
  - We can choose ξ^0 such that the trace $\phi = 0$. Note that if $\phi = 0$ then $\phi_{\mu\nu} = h_{\mu\nu}$.
  - The 3 functions ξ_i can be chosen so that $\phi^{0i} = 0$.
  - Then the Hilbert condition for $\mu = 0$ will be written $\phi_{00}{}^{,0} + \phi_{0i}{}^{,i} = 0$. But since we fixed $\phi^{0i} = 0$ we get $\phi_{00}{}^{,0} = 0$, i.e. ϕ_{00} is a constant in time.
  - A **time-independent part** term ϕ_{00} corresponds to the static part of the grav. interactions i.e. to the Newtonian potential of the source.
  - The GW itself is the **time-dependent part** and therefore as far as the GW concerns $h_{00}{}^{,0} = 0$ means $h_{00} = 0$.

In conclusion, we set

$$h^{0\mu} = 0\,, \quad h^i{}_i = 0\,, \quad h^{ij}{}_{,i} = 0 \tag{42}$$

GW: Properties

Equation (33) is the basis for computing the generation of GWs within the linearised theory.
To study the propagation of GWs as well as the interaction with test masses (and therefore the GW detector) we are interested for the equations outside the source, i.e. where $T_{\mu\nu} = 0$.
GWs are periodic changes of spacetime curvature and for weak gravitational fields far away from sources they described by a simple wave equations which admits a solution of the form:

$$\phi_{\mu\nu} = A_{\mu\nu} \cos\left(k_\alpha x^\alpha\right) \ , \tag{43}$$

where $A_{\mu\nu}$ is a symmetric tensor called polarization tensor including information of the amplitude and the polarization properties of the GWs.
$k^\alpha \equiv (k^0 = \omega/c, \vec{k})$ is the wave-vector.
This solution satisfies Hilbert's gauge condition, that is:

$$0 = {\phi_{\mu\nu}}^{,\nu} = -A_{\mu\nu} k^\nu \sin\left(k_\alpha x^\alpha\right)$$

which lead to the orthogonality condition

$$A_{\mu\nu} k^\nu = 0\,. \tag{44}$$

From the wave equation (33) we get

$$0 = \phi_{\mu\nu}{}^{,\alpha}{}_{,\alpha} = -A_{\mu\nu} k^{\alpha} k_{\alpha} \cos\left(k_{\alpha} x^{\alpha}\right) \quad \Rightarrow \quad k^{\alpha} k_{\alpha} = 0. \qquad (45)$$

This relation suggests that the wave vector k^{α} is null i.e. gravitational waves are propagating with the speed of light. But, (45) implies that $\omega^2 = c^2 |\vec{k}|^2$ i.e. both group and phase velocity of GWs are equal to the speed of light.
$v_{\rm group} = \frac{\partial \omega}{\partial k}$ and $v_{\rm phase} = \frac{\lambda}{T} = \frac{\omega}{k}$

GW: The Transverse - Traceless (TT) Gauge

Based on the gauge freedom which allows to choose ξ^μ we derived the following relations

$$h^{0\mu} = 0\,, \quad h^i{}_i = 0\,, \quad h^{ij}{}_{,i} = 0 \tag{46}$$

which define the so-called **Transverse - Traceless (TT) Gauge**.
Then for a GW propagating in the z direction i.e. it has a wave vector of the form $k_\mu = (\omega/c, 0, 0, -\omega/c)$ where $k_0 = \omega/c$ is the frequency of the wave that:

$$h^{\mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - z/c)] \tag{47}$$

While h_+ and $h_\times$, are the amplitudes of the gravitational waves in the two polarizations.
The GWs described in this spacific gauge are **T**ransverse and **T**raceless, and we will use the notation $h^{TT}_{\mu\nu}$.

We will study the effect of GWs on particles.
A static or slowly moving particle has velocity vector $u^\mu \approx (1,0,0,0)$ and one can assume that $\tau \approx t$. Then in linearized gravity the geodesic equation will be written as:

$$\frac{du^\mu}{dt} = -\frac{1}{2}\left(h_{\mu\alpha,\beta} + h_{\beta\mu,\alpha} - h_{\alpha\beta,\mu}\right) u^\alpha u^\beta \tag{48}$$

leading to

$$\frac{du^\mu}{dt} = -\left(h_{\mu 0,0} - \frac{1}{2} h_{00,\mu}\right) . \tag{49}$$

If we now use the T-T gauge ($h_{00} = h_{\mu 0} = 0$) we conclude that GWs do not affect isolated particles!
If instead we consider a pair of test particles on the cartesian axis Ox being at distances x_0 and $-x_0$ from the origin and we assume a GW traveling in the z-direction then their distance will be given by the relation:

$$\begin{aligned} d\ell^2 &= g_{\mu\nu} dx^\mu dx^\nu = \dots \\ &= -g_{11}(dx)^2 = (1 - h_{11})(2x_0)^2 = (1 - h_+ \cos\omega t)(2x_0)^2 \end{aligned} \tag{50}$$

or approximately

$$\Delta\ell \approx \left(1 - \frac{1}{2} h_+ \cos\omega t\right)(2x_0) . \tag{51}$$

GW: Effects...

In a similar way we can show for two particles on the Oy axis that:

$$\Delta\ell \approx \left(1 + \frac{1}{2} h_+ \cos\omega t\right) (2y_0) \,. \tag{52}$$

In other wards the coordinate distance of two particles is varying periodically with the time

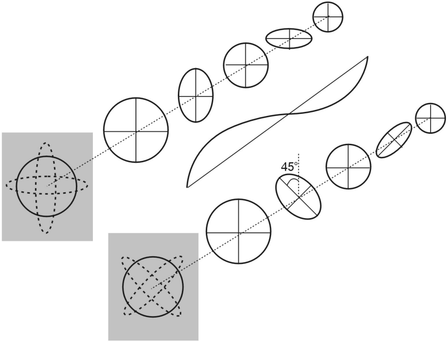


Figure: The effect of a travelling GW on a ring of particles

GW: Effects...

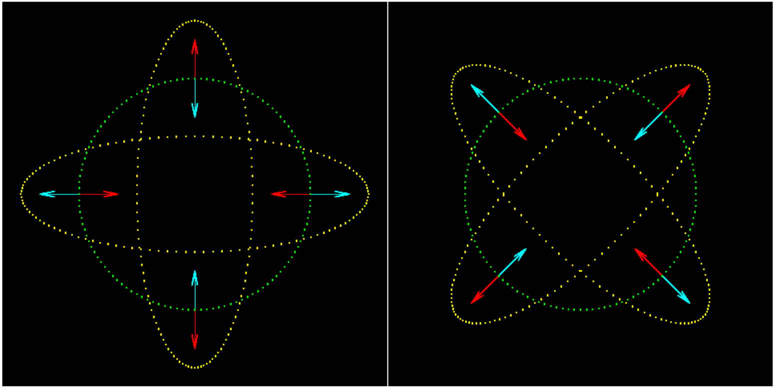


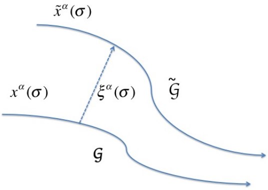
Figure: The effect of a travelling GW on a ring of particles

Geodesic deviation (*)

In a curved spacetime two geodesics that can be "parallel" initially will either converge or diverge depending on the local curvature. Consider two neighbouring geodesics $\mathcal{G}$ given by $x^\alpha(\sigma)$ and $\tilde{\mathcal{G}}$ given by $\tilde{x}^\alpha(\sigma)$ where σ is an affine parameter. If $\xi^\alpha(\sigma)$ is a small vector connecting points of the two geodesics for the same values of σ i.e.

$$\tilde{x}^\alpha(\sigma) = x^\alpha(\sigma) + \xi^\alpha(\sigma)$$

If we construct **local geodesic coordinates** about the point P, the Christoffel symbols will vanish but its derivatives will be non-zero there.



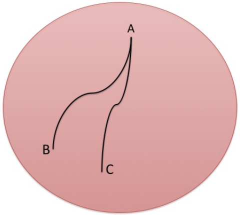


Figure: (Left) Two neighbouring geodesics. (Right) Converging geodesics on the surface of a sphere.

In this coordinate system we will get

$$\left(\frac{d^2x^\alpha}{d\sigma^2}\right)_P = 0 \quad , \quad \left(\frac{d^2\tilde{x}^\alpha}{d\sigma^2} + \Gamma^\alpha{}_{\mu\nu}\frac{d\tilde{x}^\mu}{d\sigma}\frac{d\tilde{x}^\nu}{d\sigma}\right)_Q = 0 \tag{53}$$

But since ξ^α is small:

$$[\Gamma^\alpha{}_{\mu\nu}]_Q = [\Gamma^\alpha{}_{\mu\nu}]_P + [\Gamma^\alpha{}_{\mu\nu,\lambda}]_P\,\xi^\lambda = [\Gamma^\alpha{}_{\mu\nu,\lambda}]_P\,\xi^\lambda$$

by subtracting the two equations in (53) we get (to 1st order, at P):

$$\frac{d^2\xi^\alpha}{d\sigma^2} + \Gamma^\alpha{}_{\mu\nu,\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\xi^\lambda = 0$$

However, in our geodesic coordinates the 2nd order absolute (intrinsic) derivative of ξ^α at P is:

$$\frac{D^2\xi^\alpha}{D\sigma^2} = \frac{d}{d\sigma}\left(\frac{d\xi^\alpha}{d\sigma} + \Gamma^\alpha{}_{\mu\nu}\xi^\mu x^\nu\right) = \frac{d^2\xi^\alpha}{d\sigma^2} + \Gamma^\alpha{}_{\mu\nu,\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\lambda}{d\sigma}\xi^\nu$$

where we have used the fact that $\Gamma^\alpha_{\mu\nu}(P) = 0$.
By combining the last two equations we get:

$$\frac{D^2\xi^\alpha}{D\sigma^2} + [\Gamma^\alpha{}_{\mu\lambda,\nu} - \Gamma^\alpha{}_{\mu\nu,\lambda}]\xi^\nu\frac{dx^\mu}{d\sigma}\frac{dx^\lambda}{d\sigma} = 0$$

which will give

$$\frac{D^2\xi^\alpha}{D\sigma^2} + R^\alpha{}_{\mu\nu\lambda}\xi^\nu \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = 0 \tag{54}$$

because the term in the square brackets is the Riemann tensor in local geodesic coordinates.

Tidal forces in a curved spacetime

Tidal forces deform the shape of bodies as they freely move in a gravitational field.

Thus two nearby particles with trajectories $x^i(t)$ and $\tilde{x}^i(t)$ (in Cartesian coordinates) will be separated by a vector $\xi^i = x^i - \tilde{x}^i$

$$\frac{d^2\xi}{dt^2} = -\left(\frac{\partial^2\Phi}{\partial x^i \partial x^j}\right)\xi^j \tag{55}$$

(why?) where Φ is the Newtonian gravitational potential.

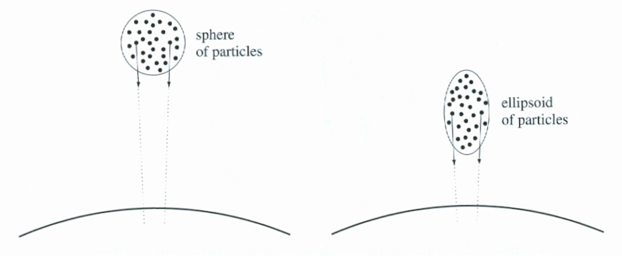


Figure: Tidal effects on a cloud of particles

Tidal effects can be also estimated in GR for two particles moving along timelike geodesics $x^{\mu}(\tau)$ and $\tilde{x}^{\mu}(\tau)$ (τ is the proper time of the 1st particle). The separation vector between the worldlines of the 2 particles is $\xi^{\mu}(\tau) = \tilde{x}^{\mu} - x^{\mu}$:

$$\frac{D^2\xi^{\mu}}{D\tau^2} = R^{\mu}{}_{\sigma\rho\nu} u^{\sigma} u^{\rho} \xi^{\nu} \equiv S^{\mu}{}_{\nu}\xi^{\nu} \tag{56}$$

where $S^{\mu}{}_{\nu}$ is the so called **tidal stress tensor** and $u^{\sigma} = du^{\sigma}/d\tau$. This is a fully covariant tensor equation and holds in any coordinate system.

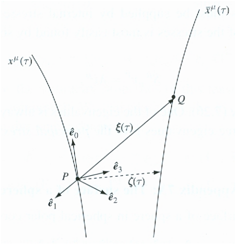


Figure: The basis vectors of the instantaneous rest frame (IRF) at P.

- $\hat{e}_\alpha$ is a set of orthonormal basis vectors at P that define the IRF of the first particle (observer) with $\hat{e}_\alpha \cdot \hat{e}_\beta = \eta_{\alpha\beta}$.
- ξ is a general connecting vector with $\xi^{\hat{\alpha}} \equiv \hat{e}^\alpha \cdot \xi = \left(\hat{e}^\alpha\right)_\mu \xi^\mu$
- ζ is the orthogonal connecting vector.

For an observer sitting on the one of the particles it can be shown that in any orthonormal freely falling frame becomes:

$$\frac{d^2\xi^{\hat{\mu}}}{d\tau^2} = c^2 R^{\hat{\mu}}{}_{\hat{0}\hat{0}\hat{\nu}} \xi^{\hat{\nu}} . \tag{57}$$

Newtonian limit (we will discuss the details later)

$$R^{\hat{\mu}}{}_{\hat{0}\hat{0}\hat{\nu}} \to \frac{\partial^2 \Phi}{\partial x^i \partial x^j} \tag{58}$$

GW: Tidal forces

Riemann tensor is a **"measure" of spacetime's curvature** and in linearized gravity gets the form

$$R_{\kappa\lambda\mu\nu} = \frac{1}{2}\left(\partial_{\nu\kappa} h_{\lambda\mu} + \partial_{\lambda\mu} h_{\kappa\nu} - \partial_{\kappa\mu} h_{\lambda\nu} - \partial_{\lambda\nu} h_{\kappa\mu}\right), \qquad (59)$$

in the T-T gauge the Riemann tensor is considerably simplified

$$R^{\mathrm{TT}}_{j0k0} = -\frac{1}{2}\frac{\partial^2}{\partial t^2} h^{\mathrm{TT}}_{jk}, \quad \text{for} \quad j,k = 1,2,3. \qquad (60)$$

Actually, the Newtonian limit of the Riemann tensor is:

$$R^{\mathrm{TT}}_{j0k0} \approx \frac{\partial^2 U}{\partial x^j \partial x^k}, \qquad (61)$$

where U is the Newtonian potential. In other words the Riemann tensor has also a pure physical meaning i.e. it is a measure of the tidal gravitational acceleration. Then the distance between two nearby particles $x^\mu(\tau)$ will $x^\mu(\tau) + \xi^\mu(\tau)$ will be described by

$$\frac{d^2\xi^k}{dt^2} \approx -{R^k}_{0j0}{}^{\mathrm{TT}} \xi^j. \qquad (62)$$

The tidal force acting on a particle is (why?)

$$f^k \approx -mR^k{}_{0j0}\xi^j \approx \frac{m}{2}\frac{d^2 h_{jk}^{TT}}{dt^2}\xi^j \tag{63}$$

where m is particle's mass. This means that

$$f^x \approx \frac{m}{2} h_+ \omega^2 \cos[\omega(t-z)]\xi_0^x, \quad \text{and} \quad f^y \approx -\frac{m}{2} h_+ \omega^2 \cos[\omega(t-z)]\xi_0^y. \tag{64}$$

$$\nabla \vec{f} = \frac{\partial f^x}{\partial \xi_0^x} + \frac{\partial f^y}{\partial \xi_0^y} = 0\,. \tag{65}$$

Hence the divergence of the force $\vec{f}$ is zero, which tell us that the tidal force can be represented graphically by field lines.

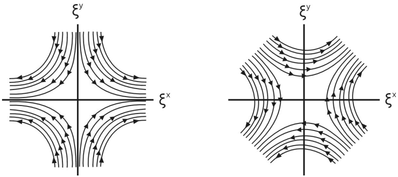


Figure: The tidal field lines of force for a gravitational wave with polarization $(+)$ (left panel) and $(\times)$ (right panel). The orientation of the field lines changes every half period producing the deformations as seen in Figure 1. Any point accelerates in the directions of the arrows, and the denser are the lines, the strongest is the acceleration. Since the acceleration is proportional to the distance from the center of mass, the force lines get denser as one moves away from the origin. For the polarization $(\times)$ the force lines undergo a 45^0 rotation.

GW: Properties

- GWs, once they are generated, propagate almost unimpeded. Indeed, they are even harder to stop than neutrinos! The only significant change they suffer as they propagate is the decrease in amplitude while they travel away from their source, and the *redshift* they feel (cosmological, gravitational or Doppler).

- EM waves are fundamentally different, however, even though they share similar wave properties away from the source.

- GWs are emitted by coherent bulk motions of matter (for example, by the implosion of the core of a star during a supernova explosion) or by coherent oscillations of spacetime curvature, and thus they serve as a probe of such phenomena.

By contrast,

- Cosmic EM waves are mainly the result of incoherent radiation by individual atoms or charged particles.

⋆ As a consequence, from the cosmic electromagnetic radiation we mainly learn about the form of matter in various regions of the universe, especially about its temperature and density, or about the existence of magnetic fields.

GW: Properties

- Strong GWs are emitted from regions of spacetime where gravity is very strong and the velocities of the bulk motions of matter are near the speed of light.

Since most of the time these areas are either surrounded by thick layers of matter that absorb EM radiation or they do not emit any at all (black holes), the only way to study these regions of the universe is via GWs.

GW: The energy of GWs

The fact that GWs carry energy and momentum is already clear from the discussion on the interaction with test masses.
To get an explicit expression of the energy-momentum tensor of GWs we can follow two different routes one more geometrical and the other more field-theoretical

A. According to GR, any form of energy contributes to the curvature of space-time, thus we can ask "whether GWs are themselves a source of space-time curvature".

B. We can treat linearised gravity as any other classical filed theory and apply Noether's theorem, the standard field-theoretical tool that answers this question.

GW: The energy of GWs

In order to include the contribution of the energy-momentum associated with the gravitational field itself one must modify the linearise Einstein's equations to read

$$G^{(1)}_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} + \mathcal{T}_{\mu\nu}\right) \tag{66}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of any matter present and $\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor of the gravitational field itself.
On the other hand Einstein's equations may expand beyond first order to obtain

$$G_{\mu\nu} \equiv G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + \cdots = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{67}$$

This suggest that, to a good approximation, we should make the identification

$$\mathcal{T}_{\mu\nu} \equiv \frac{c^4}{8\pi G} G^{(2)}_{\mu\nu} = \ldots = \frac{c^4}{8\pi G}\left\langle G^{(2)}_{\mu\nu}\right\rangle \tag{68}$$

GW: Energy

GWs carry energy. The stress-energy carried by GWs cannot be localized within a wavelength. Instead, one can say that a certain amount of stress-energy is contained in a region of the space which extends over several wavelengths. The stress-energy tensor can be written as:

$$\mathcal{T}^{\mu\nu} = \frac{1}{4}\left[2\phi^{\alpha\beta,\mu}\phi_{\alpha\beta}{}^{,\nu} - \phi^{,\mu}\phi^{,\nu} - \eta^{\mu\nu}\left(\phi^{\alpha\beta,\sigma}\phi_{\alpha\beta,\sigma} - \frac{1}{2}\phi_{,\sigma}\phi^{,\sigma}\right)\right] \tag{69}$$

which in the TT gauge of the linearized theory becomes (HOW?)

$$\mathcal{T}^{GW}_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle\left(\partial_\mu h_{ij}^{TT}\right)\left(\partial_\nu h_{ij}^{TT}\right)\right\rangle . \tag{70}$$

where the angular brackets indicate averaging over several wavelengths. For the special case of a plane wave propagating in the z direction, the stress-energy tensor has only three non-zero components, which take the simple form

$$\mathcal{T}^{GW}_{00} = \frac{\mathcal{T}^{GW}_{zz}}{c^2} = -\frac{\mathcal{T}^{GW}_{0z}}{c} = \frac{1}{32\pi}\frac{c^2}{G}\omega^2\left(h_+^2 + h_\times^2\right), \tag{71}$$

where $\mathcal{T}^{GW}_{00}$ is the energy density, $\mathcal{T}^{GW}_{zz}$ is the momentum flux and $\mathcal{T}^{GW}_{0z}$ the energy flow along the z direction per unit area and unit time .

Gravitational Waves: Nature of

$\star$ Electromagnetic radiation emitted by slowly varying charge distributions can be decomposed into a series of multipoles, where the amplitude of the 2^ℓ-pole $(\ell = 0, 1, 2, ...)$ contains a small factor a^ℓ, with a equal to the ratio of the diameter of the source to the typical wavelength, namely, a number typically much smaller than 1.

From this point of view the strongest EM radiation would be expected for monopolar radiation ($\ell = 0$), but this is completely absent, because the EM monopole moment is proportional to the total charge, which does not change with time (*it is a conserved quantity*).

Therefore, EM radiation consists only of $\ell \geq 1$ multipoles, the strongest being the electric dipole radiation ($\ell = 1$), followed by the weaker magnetic dipole & electric quadrupole radiation ($\ell = 2$).

Gravitational Waves: Nature of

For Gravitational Waves, it can be shown that **mass conservation** (which is equivalent to charge conservation in EM theory) will exclude monopole radiation.

Also, the rate of change of the mass dipole moment is proportional to the **linear momentum** of the system, which is a conserved quantity, and therefore there cannot be any mass dipole radiation in Einstein's relativity theory.

The next strongest form of EM radiation is the magnetic dipole. For the case of gravity, the change of the "magnetic dipole" is proportional to the **angular momentum** of the system, which is also a conserved quantity and thus there is no dipolar grav. radiation of *any* sort. It follows that grav. radiation is of quadrupolar (or higher nature) and is directly linked to the quadrupole moment of the mass distribution.

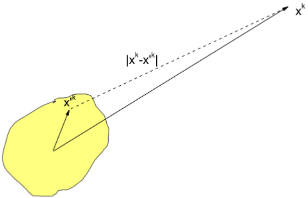
Newtonian Gravity

Poisson equation

$$\nabla^2 U(\vec{x}) = 4\pi G\rho(\vec{x}) \rightarrow U(\vec{x}) = -G\int d^3\vec{x'}\,\frac{\rho(\vec{x})}{|\vec{x}-\vec{x'}|}$$

For a spherically symmetric mass distribution of radius R

$$\begin{aligned} U(r) &= -\frac{1}{r}\int_0^R r'^2\rho(r')dr' \quad \text{for} \quad r > R \\ U(r) &= -\frac{1}{r}\int_0^r r'^2\rho(r')dr' - \int_r^R r'\rho(r')dr' \quad \text{for} \quad r < R \end{aligned}$$



For a non-spherical distribution the term $1/|\vec{x}-\vec{x'}|$ can be expanded as

$$\frac{1}{|\vec{x}-\vec{x'}|} = \frac{1}{r} + \sum_k \frac{x^k x'^k}{r^3} + \frac{1}{2}\sum_k \sum_l \left(3x'^k x'^l - \vec{r}'^2 \delta_k^l\right) \frac{x^k x^l}{r^5} + \dots$$

$$U(\vec{x}) = -\frac{GM}{r} - \frac{G}{r^3}\sum_k x^k D^k - \frac{G}{2}\sum_{kl} Q^{kl}\frac{x^k x^l}{r^5} + \dots$$

Gravitational Multipoles

$$\begin{aligned} M &= \int \rho(\vec{x'}) d^3x' \quad \text{Mass} \\ D^k &= \int x'^k \rho(\vec{x'}) d^3x' \quad \text{Mass Dipole moment}^2 \\ Q^{kl} &= \int \left(3x'^k x'^l - \vec{r'}^2 \delta_k^l\right) \rho(\vec{x'}) d^3x' \quad \text{Mass Quadrupole tensor}^3 \end{aligned}$$

[2] If the center of mas is chosen to coincide with the origin of the coordinates then $D^k = 0$ (no mass dipole).

[3] If $Q^{kl} \neq 0$ the potential will contain a term proportional to $\sim 1/r^3$ and the gravitational force will deviate from the inverse square law by a term $\sim 1/r^4$.

GW: Generation

Einstein (1918) derived the quadrupole formula for gravitational radiation by solving the linearized form of his equations

$$\Box\phi^{\mu\nu}(t,\vec{x}) = -\kappa T^{\mu\nu}(t,\vec{x})\,. \tag{72}$$

The solution is:

$$\phi^{\mu\nu}(t,\vec{x}) = -\frac{\kappa}{4\pi}\int_V \frac{T^{\mu\nu}\left(t-|\vec{x}-\vec{x}'|,\vec{x}'\right)}{|\vec{x}-\vec{x}'|}d^3x'\,, \tag{73}$$

This solution suggests that ϕ_{ij} is proportional to the second time derivative of the quadrupole moment of the source (WHY?):

$$\phi_{ij} = \frac{2}{r}\frac{G}{c^4}\ddot{Q}_{ij}^{TT}(t-r/c) \quad \text{where} \quad Q_{ij}^{TT}(x) = \int \rho\left(x^i x^j - \frac{1}{3}\delta^{ij}r^2\right)d^3x \tag{74}$$

where, Q_{ij}^{TT} is the quadrupole moment in the TT gauge, evaluated at the retarded time $t-r/c$ [4].

The **energy** radiated by the system per unit solid angle and unit time in the direction n^s is

$$-\frac{d^2E}{dtd\Omega} = r^2\mathcal{T}^{0s}n^s \tag{75}$$

[4] This result is quite accurate for all sources, as long as the reduced wavelength $\tilde{\lambda} = \lambda/2\pi$ is much longer than the source size R.

GW: Emission of Energy, Angular and Linear Momentum

Using the formulae (70) and (71) for the energy carried by GWs, one can derive the luminosity in GWs as a function of the third-order time derivative of the quadrupole moment tensor.
This is the well-known quadrupole formula for the **Energy emission**

$$L_{GW} = -\frac{dE}{dt} = \frac{1}{5}\frac{G}{c^5}\left\langle \dddot{Q}_{ij} \cdot \dddot{Q}_{ij} \right\rangle \tag{76}$$

Angular momentum emission

$$\frac{dJ_i^{\mathrm{GW}}}{dt} = \frac{2}{5}\sum_{jk\ell} \epsilon_{ijk}\left\langle \ddot{Q}_{j\ell} \cdot \dddot{Q}_{\ell k} \right\rangle \tag{77}$$

Linear momentum emission

$$\frac{dP_i^{\mathrm{GW}}}{dt} = \frac{2}{63}\sum_{jk}\left\langle \dddot{Q}_{jk} \cdot \dddot{Q}_{jki} \right\rangle + \frac{16}{45}\sum_{jk\ell} \epsilon_{ijk}\left\langle \dddot{Q}_{j\ell} \cdot \dddot{P}_{\ell k} \right\rangle \tag{78}$$

where
Q_{ijk} : mass octupole moment
P_{ij} : current quadrupole moment

GW: Energy Flux

The energy flux has all the properties one would anticipate by analogy with electromagnetic waves: (a) it is conserved (the amplitude dies out as $1/r$, the flux as $1/r^2$), (b) it can be absorbed by detectors, and (c) it can generate curvature like any other energy source in Einstein's formulation of relativity.

Estimate the energy flux in GWs from the collapse of the core of a supernova to create a 10 $M_\odot$ black hole at a distance of $\sim$15 Mpc from the earth (at the distance of the Virgo cluster of galaxies). An optimistic estimate of the amplitude of the GWs on Earth is of the order of $h \approx 10^{-22}$ (at a frequency of about 1kHz). This corresponds to a flux of about 3 ergs/cm^2 sec. This is an enormous amount of energy flux and is about ten orders of magnitude larger than the observed energy flux in electromagnetic waves!

The basic difference is the duration of the two signals; GW signal will last a few milliseconds, whereas an EM signal lasts many days. This example provides us with a useful numerical formula for the energy flux:

$$F = 3\left(\frac{f}{1\mathrm{kHz}}\right)^2\left(\frac{h}{10^{-22}}\right)^2 \frac{\mathrm{ergs}}{\mathrm{cm}^2\mathrm{sec}}, \tag{79}$$

from which one can easily estimate the flux on Earth, given the amplitude (on Earth) and the frequency of the waves.

GW: Order of magnitude estimates

The quadrupole moment of a system is approximately equal to the mass M of the part of the system that moves, times the square of the size R of the system. This means that the 3rd-order time derivative of the quadrupole moment is

$$\frac{\partial^3 Q_{ij}}{\partial t^3} \sim \frac{MR^2}{T^3} \sim \frac{M\mathcal{V}^2}{T} \sim \frac{E_{\rm ns}}{T}, \tag{80}$$

where $\mathcal{V}$ is the mean velocity of the moving parts, $E_{\rm ns}$ is the kinetic energy of the component of the source's internal motion which is non-spherical, and T is the time scale for a mass to move from one side of the system to the other. The time scale (or period) is actually proportional to the inverse of the square root of the mean density of the system (why?)

$$T \sim \sqrt{R^3/GM}. \tag{81}$$

This relation provides a rough estimate of the characteristic frequency of the system $f = 2\pi/T$. The luminosity of GWs of a given source is approximately

$$L_{GW} \sim \frac{G^4}{c^5}\left(\frac{M}{R}\right)^5 \sim \frac{G}{c^5}\left(\frac{M}{R}\right)^2 \mathcal{V}^6 \sim \frac{c^5}{G}\left(\frac{R_{\rm Sch}}{R}\right)^2\left(\frac{\mathcal{V}}{c}\right)^6 \tag{82}$$

where $R_{\rm Sch} = 2GM/c^2$ is the Schwarzschild radius of the source. It is obvious that the maximum value of the luminosity in GWs can be achieved if the source's dimensions are of the order of its Schwarzschild radius and the typical velocities of the components of the system are of the order of the speed of light.

GW: Order of magnitude estimates II

The above formula sets also an upper limit on the power emitted by a source, which for $R \sim R_{\rm Sch}$ and $\mathcal{V} \sim c$ is:

$$L_{GW} \sim c^5/G = 3.6 \times 10^{59} \mathrm{ergs/sec}. \tag{83}$$

This is an immense amount of power, often called the *luminosity of the universe*.

Using the above order-of-magnitude estimates, we can get a rough estimate of the amplitude of GWs at a distance r from the source:

$$h \sim \frac{G}{c^4}\frac{E_{\rm ns}}{r} \sim \frac{G}{c^4}\frac{\varepsilon E_{\rm kin}}{r} \tag{84}$$

where $\varepsilon E_{\rm kin}$ (with $0 \leq \varepsilon \leq 1$), is the fraction of kinetic energy of the source that is able to produce GWs. The factor ε is a measure of the asymmetry of the source and implies that only a time varying quadrupole moment will emit GWs. Another formula for the amplitude of GW relation can be derived from the flux formula (79). If, for example, we consider an event (perhaps a supernovae explosion) at the Virgo cluster during which the energy equivalent of $10^{-4} M_\odot$ is released in GWs at a frequency of 1 kHz, and with signal duration of the order of 1 msec, the amplitude of the gravitational waves on Earth will be

$$h \approx 10^{-22}\left(\frac{E_{GW}}{10^{-4}M_\odot}\right)^{1/2}\left(\frac{f}{1\mathrm{kHz}}\right)^{-1}\left(\frac{\tau}{1\mathrm{msec}}\right)^{-1/2}\left(\frac{r}{15\mathrm{Mpc}}\right)^{-1}. \tag{85}$$

GW: Order of magnitude estimates III

For a detector with arm length of 4 km we are looking for changes in the arm length of the order of

$$\Delta\ell = h \cdot \ell = 10^{-22} \cdot 4\,\mathrm{km} = 4 \times 10^{-17}\mathrm{cm!!!}$$

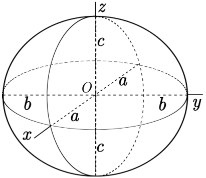
These numbers shows why experimenters are trying so hard to build ultra-sensitive detectors and explains why all detection efforts till recently were not successful.
Finally, it is useful to know the damping time, that is, the time it takes for a source to transform a fraction $1/e$ of its energy into gravitational radiation. One can obtain a rough estimate from the following formula

$$\tau = \frac{E_{\mathrm{kin}}}{L_{\mathrm{GW}}} \sim \frac{1}{c} R \left(\frac{R}{R_{\mathrm{Sch}}}\right)^3 . \qquad (86)$$

For example, for a non-radially oscillating neutron star with a mass of roughly $1.4M_{\odot}$ and a radius of 12Km, the damping time will be of the order of $\sim$50msec. Also, by using formula (81), we get an estimate for the frequency of oscillation which is directly related to the frequency of the emitted gravitational waves, roughly 2kHz for the above case.

Example: Quadrupole Moment Tensor

We will calculate the mass quadrupole moment tensor of a homogeneous triaxial ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. By setting $x' = x/a$, $y' = y/a$ and $z' = z/a$ the volume integration over the ellipsoid reduces to that over the unit sphere



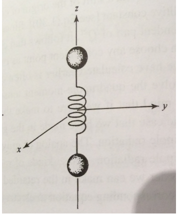
$$
\begin{aligned}
Q_{11} &= \int\int\int \rho\left(3x^2 - r^2\right) dxdydz = \int\int\int \rho\left(2x^2 - y^2 - z^2\right) dx \\
&= \int\int\int \rho abc\left(2a^2x'^2 - b^2y'^2 - c^2z'^2\right) dx'dy'dz' \\
&= \rho abc\left(2a^2 - b^2 - c^2\right)\int\int\int z'^2 dx'dy'dz' \\
&= \rho abc\left(2a^2 - b^2 - c^2\right)\int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^4 dr \cos^2\theta \sin\theta d\theta d\phi \\
&= \frac{m}{5}\left(2a^2 - b^2 - c^2\right)
\end{aligned}
$$

where $m = \frac{4}{3}\pi abc\rho$ is the mass of the ellipsoid. The other two non-vanishing components of the mass quadrupole tensor are [5]:

$$
Q_{22} = \frac{m}{5}\left(-a^2 + 2b^2 - c^2\right) \quad \text{and} \quad Q_{33} = \frac{m}{5}\left(-a^2 - b^2 + 2c^2\right)
$$

[5] By definition, the mass quad. moment tensor is traceless, $Q_{jj} = Q_{11} + Q_{22} + Q_{33}$.

Example: Vibrating Quadrupole I



The distance of the masses from the center varies periodically as $z = \pm(b + a\sin\omega t)$. The quadrupole moment tensor for the pair of equal masses m is:

$$Q(0)^{ij} \equiv \begin{pmatrix} -\frac{2}{3}mb^2 & 0 & 0 \\ 0 & -\frac{2}{3}mb^2 & 0 \\ 0 & 0 & \frac{4}{3}mb^2 \end{pmatrix} \tag{88}$$

Then the retarded value of the quadrupole tensor is:

$$Q^{ij}(t-r) \approx \left[1 + \frac{2a}{b}\sin\omega(t-r)\right] Q(0)^{ij} \tag{89}$$

The radiated gravitational field is:

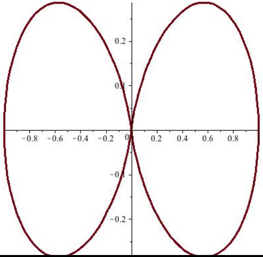
$$\phi^{ij} = \frac{2}{r}\frac{G}{c^4}\ddot{Q}^{ij}(t-r) = \frac{2}{r}\frac{G}{c^4}\frac{2a}{b}\omega^2 \sin\omega(t-r) Q(0)^{ij} \tag{90}$$

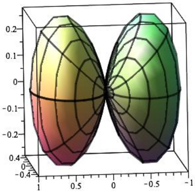
Example: Vibrating Quadrupole II

The energy radiated by the system per unit solid angle and unit time in the direction n^s is

$$\begin{aligned} -\frac{d^2E}{dtd\Omega} &= r^2\mathcal{T}^{0s}n^s \\ &= -\frac{1}{18}\left(\frac{\kappa}{8\pi}\right)^2 \mathcal{Q} = \left(\frac{\kappa}{8\pi}\right)^2 \left[2mab\omega^3\cos\omega(t-r)\right]^2 \sin^4\theta \quad (91) \end{aligned}$$

$$\begin{aligned} \mathcal{Q} = &\left(\dddot{Q}^{11}\right)^2 + \left(\dddot{Q}^{22}\right)^2 + \left(\dddot{Q}^{33}\right)^2 - 2\left(\dddot{Q}^{11}n^1\right)^2 - 2\left(\dddot{Q}^{22}n^2\right)^2 - 2\left(\dddot{Q}^{33}n^3\right)^2 \\ &+\frac{1}{2}\left(\ddot{Q}^{11}n^1n^1 + \ddot{Q}^{22}n^2n^2 + \ddot{Q}^{33}n^3n^3\right) \end{aligned}$$





Example: Vibrating Quadrupole III

The **total emitted power** is:

$$L_{GW} = -\frac{dE}{dt} = \frac{1}{5}\frac{G}{c^5}\left\langle \dddot{Q}_{ij}\cdot\dddot{Q}_{ij}\right\rangle = \frac{32}{15}\frac{G}{c^5}\left\langle mab\omega^3\cos\omega(t-r)\right\rangle^2$$
$$\approx \frac{16G}{15c^5}(mab)^2\omega^6 \qquad (92)$$

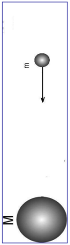
and the **damping time** of the oscillator, due to the emission of GWs is :

$$\frac{1}{\tau_{\rm rad}} = -\frac{1}{E}\frac{dE}{dt} = \frac{16}{15}\frac{G}{c^5}mb^2\omega^4 \quad \text{where} \quad E = \frac{1}{2}m\omega^2a^2 \qquad (93)$$

The above formulae give an order of magnitude estimate for the GW emission of by any vibrating elastic body, provided that the vibrations are not spherical.

Example: Two-body collision I

We assume that a particle of mass m starts from infinity with zero velocity ($\frac{1}{2}m\dot{z}^2 = \frac{GmM}{|z|}$, $\ddot{z} = GM/z^2$, $\dddot{z} = (2GM)^{3/2}/|z|^{7/2}$) falls towards a massive body of mass M .



Radiated power

$$-\frac{dE}{dt} = \frac{8}{15}\frac{G}{c^5}m^2\left(3\dot{z}\ddot{z} + z\dddot{z}\right)^2 \tag{94}$$

The **energy during the plunge** from $z = \infty$ to $z = R$

$$-\Delta E = \frac{4}{105}\frac{G}{c^5}\frac{m^2(2GM)^{5/2}}{R^{7/2}} \tag{95}$$

If $R = R_{\rm Schw}$ ($M = 10M\odot$ & $m = 1M\odot$)

$$-\Delta E = 0.019mc^2\frac{m}{M} \tag{96}$$

$$-\Delta E = 0.0104mc^2\frac{m}{M} \to 2\times 10^{51}\text{erg} \tag{97}$$

Most radiation during the $2R \to R$ phase

$$\Delta t \sim R/\nu \sim R/c \sim 30km/c \sim 10^{-4}sec \quad \to \quad f \sim 10^4 Hz \tag{98}$$

Example: Two-body collision III

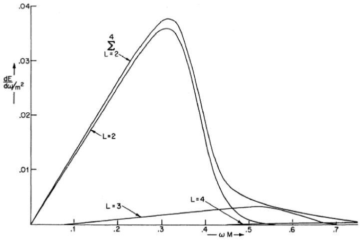


Figure: Spectrum of GW emitted by a particle of mass m falling radially into a BH of mass M. The quantity $dE/d\omega$ gives the amount of energy radiated per unit frequency interval. The curves marked $L = 2, 3, 4$ correspond to quadrupole, ... radiation. Note that most of the radiation is emitted with frequency $\omega \sim 0.3 - 0.5c^3/GM$.

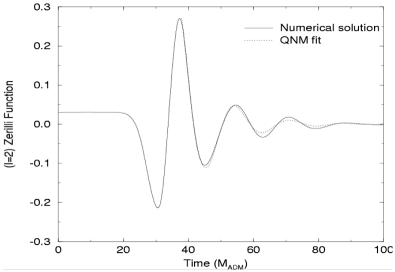
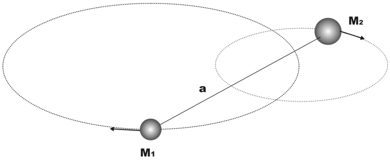


Figure: The signal of a ringing black-hole. The signal can be produced by a small body falling into a black-hole.

GW: Binaries, an example I

If we assume that the two bodies m_1 and m_2 making up the binary lie in the $x - y$ plane at distances a_1 and a_2 from the center of mass, their orbits are circular and rotating at angular frequency Ω.



Then the only non-vanishing components of the quadrupole tensor are (why?) :

$$Q_{xx} = -Q_{yy} = \left(a_1^2 M_1 + a_2^2 M_2\right) \cos^2 \Omega t = \frac{1}{2}\mu a^2 \cos 2\Omega t, \qquad (99)$$

$$Q_{xy} = Q_{yx} = \frac{1}{2}\mu a^2 \sin 2\Omega t, \qquad (100)$$

where $a = a_1 + a_2$, $a_1 M_1 = a_2 M_2 = a\mu$. Here $\mu = M_1 M_2 / M$ is the reduced mass of the system and $M = M_1 + M_2$ its total mass.

GW: Binaries, an example II

The GW luminosity of the system is (we use Kepler's third law, $\Omega^2 = GM/a^3$) (how?)

$$L^{\rm GW} = -\frac{dE}{dt} = \frac{1}{5}\frac{G}{c^5}(2\Omega)^2\left(\frac{1}{2}a^2\mu\right)^2\left\langle \sin^2 2\Omega t + \sin^2 2\Omega t + 2\cos 2\Omega t\right\rangle$$
$$= \frac{32}{5}\frac{G}{c^5}\mu^2 a^4\Omega^6 = \frac{32}{5}\frac{G^4}{c^5}\frac{M^3\mu^2}{a^5}. \qquad (101)$$

The total energy of the binary system can be written as (why?) :

$$E = \left(\frac{1}{2}M_1 a_1^2 + \frac{1}{2}M_2 a_2^2\right)\Omega^2 - \frac{GM_1 M_2}{a} = -\frac{1}{2}\frac{G\mu M}{a} \qquad (102)$$

GW: Binaries, an example III

As the gravitating system loses energy by emitting radiation, the distance between the two bodies shrinks at a rate

$$\frac{dE}{dt} = \frac{1}{2}\frac{G\mu M}{a^2}\frac{da}{dt} \quad \Rightarrow \frac{da}{dt} = -\frac{64}{5}\frac{G^3}{c^5}\frac{\mu M^2}{a^3}, \tag{103}$$

and the orbital frequency increases accordingly ($\dot{f}/f = (3/2)\dot{a}/a$).
If, the present separation of the two stars is a_0, then the binary system will coalesce after a time

$$\tau = \frac{5}{256}\frac{c^5}{G^3}\frac{a_0^4}{\mu M^4} \tag{104}$$

Finally, the amplitude of the GWs is (why?)

$$h = 5 \times 10^{-22}\left(\frac{M}{2.8M_\odot}\right)^{2/3}\left(\frac{\mu}{0.7M_\odot}\right)\left(\frac{f}{100\mathrm{Hz}}\right)^{2/3}\left(\frac{15\mathrm{Mpc}}{r}\right). \tag{105}$$

NOTE: if $m_1 = m_2 = 30M\odot$ then the amplitude will be higher by about **160 times** from a binary with $m_1 = m_2 = 1.4M\odot$!
In all these formulae we have assumed that the orbits are circular.
In general, the orbits of the two bodies are approximately ellipses, but it has been shown that long before the coalescence of the two bodies, **the orbits become circular**, at least for long-lived binaries, due to gravitational radiation.

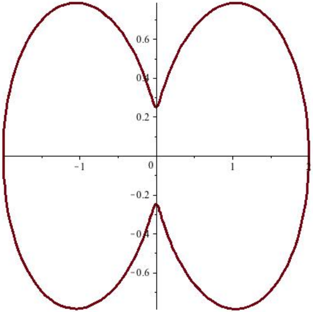


Figure: The function $g(\theta)$ in polar coordinates

The angular distribution of the radiated power, is given by

$$\left(\frac{dP}{d\Omega}\right) = \frac{r^2 c^3}{16\pi G}\left\langle \dot{h}_+^2 + \dot{h}_\times \right\rangle \tag{106}$$

or

$$\left(\frac{dP}{d\Omega}\right) = \frac{2G\mu^2 a^2 \omega^6}{\pi c^5} g(\theta) \tag{107}$$

$$g(\theta) = \left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta \tag{108}$$

GW: Binaries, an example V

- The amplitude of the emitted GWs depends on the angle between the line of sight and the axis of angular momentum; formula (105) refers to an observer along the axis of the orbital angular momentum.
- The complete formula for the amplitude contains angular factors of order 1. The relative strength of the two polarizations depends on that angle as well.
- If 3 or more detectors observe the same signal it is possible to reconstruct the full waveform and deduce many details of the orbit of the binary system.
- As an example, we will provide some details of the well-studied pulsar PSR 1913+16 (the Hulse-Taylor pulsar), which is expected to coalesce after $\sim 3.5 \times 10^8$ years. The binary system is roughly 5kpc away from Earth, the masses of the two neutron stars are estimated to be $\sim 1.4 M_\odot$ each, and the present period of the system is ~7h and 45min. The predicted rate of period change is $\dot{T} = -2.4 \times 10^{-12} \mathrm{sec/sec}$, while the corresponding observed value is in excellent agreement with the predictions, i.e., $\dot{T} = (-2.30 \pm 0.22) \times 10^{-12} \mathrm{sec/sec}$; finally the present amplitude of gravitational waves is of the order of $h \sim 10^{-23}$ at a frequency of $\sim 7 \times 10^{-5}\mathrm{Hz}$.

GW: Basic Formulae

For a gravitational wave binary with masses m_1 and m_2, in a circular orbit with gravitational wave frequency $f = 2\Omega = 2\sqrt{GM/a^3}$, then:

$$\mathcal{M}_c = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} \equiv \mu^{3/5} M^{2/5} \qquad \text{chirp mass} \tag{109}$$

$$h_o = \frac{4G}{c^5}\frac{\mathcal{M}_c}{D}\left(\frac{G}{c^3}\pi f \mathcal{M}_c\right)^{2/3} \qquad \text{scaling amplitude} \tag{110}$$

$$\dot{f} = \frac{96}{5}\frac{c^3}{G}\frac{f}{\mathcal{M}_c}\left(\frac{G}{c^3}\pi f \mathcal{M}_c\right)^{8/3} \qquad \text{chirp} \tag{111}$$

The **chirp** indicates that as GWs are emitted, they carry energy away from the binary. The gravitational binding energy decreases, and the orbital frequency increases.

The GW **phase** $\phi(t)$ evolves in time as

$$\phi(t) = 2\pi f\left(t + \frac{1}{2}\frac{\dot{f}}{f}t^2\right) + \phi_0 \tag{112}$$

where $\dot{F}$ is the chirp given above, and ϕ_0 is the initial phase of the binary.

GW: Basic Formulae

A phenomenological form of the waveform then is given by

$$h(t) = h_o \cos(\phi t) = h_o \cos\left(2\pi f t + \pi \dot{f} t^2 + \phi_0\right) \tag{113}$$

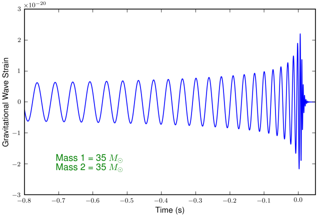


Figure: Chirp waveform for merging black-holes with masses $m_1 = m_2 = 35 M_\odot$.

This is called a **chirp** or a **chirp waveform**, characterized by an increase in amplitude and frequency as time increases.

GW: Basic Formulae

Luminosity Distance from Chirping Binaries

Suppose we can measure the chirp $\dot{f}$ and the gravitational wave amplitude h_o.
The chirp can be inverted to give the chirp mass:

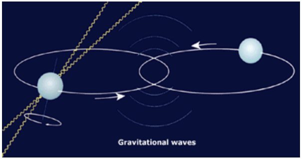
$$\mathcal{M}_c = \frac{c^3}{G}\left(\frac{5}{96}\pi^{-8/3} f^{-11/3}\dot{f}\right)^{3/5} \tag{114}$$

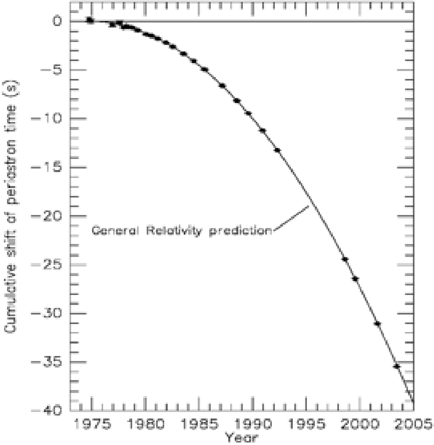
If this chirp mass is used in the amplitude equation, one can solve for the **luminosity distance** D:

$$D = \frac{5}{96\pi^2}\frac{c}{h_o}\frac{\dot{f}}{f^3} = \frac{1}{h_o}\frac{(G\mathcal{M}_c)^{5/3}}{c^4}(\pi f)^{2/3} \tag{115}$$

This is a method of measuring the luminosity distance using **only gravitational wave observables**!
This is extremely useful as an independent distance indicator in astronomy.
PROBLEM: What is the distance for a binary system with $m_1 = m_2 = 35 M_\odot$ and $S/N = 20$ for Advance LIGO?

GW: Binaries, an example: PSR 1913+16





Hulse & Taylor : Nobel 1993

GW: Known Binary Systems as Sources of GWs (*)

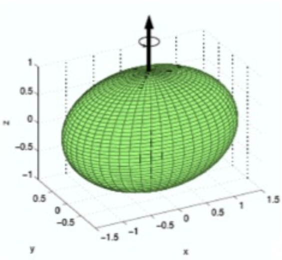
System	Masses $M_\odot$	Distances pc	Frequency 10^{-6} Hz	Luminosity 10^{30} erg/s	Amplitude 10^{-22}
[6] ι Boo	(1.0, 0.5)	11.7	86	1.1	51
μ Sco	(12, 12)	109	16	51	210
[7] Am CVn	(1.0, 0.041)	100	1900	300	5
WZ Sge	(1.5, 0.12)	75	410	24	8
[8] Cyg X-1	(19,15)	1800	4.1	2.6	9
PSR 1913+16	(1.4,1.4)	5000	70	0.6	0.12

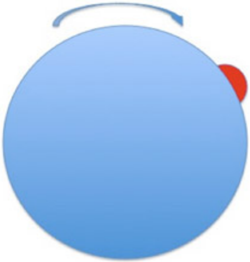
[6] Eclisping Binaries
[7] Cataclysmic Binaries
[8] Binary X-ray sources

GW Sources: "Mountains"

Axisymmetric bodies rotating about their symmetry axis have no time varying quadrupole moment and hence they do not radiate GWs.
Radiation will be produced:

- If it rotates about the principal axis and is non-axisymmetric
- If it is axisymmetric but the rotation axis is not the symmetry axis.





If I_1, I_2 and I_3 are the principal moments of inertia then we will consider the first case i.e. when $I_1 \neq I_2$.
A possible astrophysical application would be a pulsar where the rigid crust supports a "mountain".

GW Sources: "Mountains"

Applying the quadrupole formula we can get ($\phi = \Omega t$)

$$I_{xx} = \cos^2\phi I_1 + \sin^2\phi I_2 = \frac{1}{2}\cos 2\phi(I_1 - I_2) + \text{const} \quad (116)$$

$$I_{xy} = I_{yx} = \frac{1}{2}\sin 2\phi(I_1 - I_2) \quad (117)$$

$$I_{yy} = \frac{1}{2}\cos 2\phi(I_2 - I_1) + \text{const} \quad (118)$$

$$I_{zz} = \text{const}, \quad I_{xz} = I_{yz} = 0 \quad (119)$$

$$I_{ij} = -Q_{ij} + \frac{1}{3}\delta_{ij}\mathrm{Tr}Q \quad (120)$$

Thus

$$\frac{dE}{dt} = -\frac{1}{5}\frac{G}{c^5}\left\langle \left(\frac{d^3 I_{xx}}{dt^3}\right)^2 + 2\left(\frac{d^3 I_{xy}}{dt^3}\right)^2 + \left(\frac{d^3 I_{yy}}{dt^3}\right)^2 \right\rangle \quad (121)$$

$$= -\frac{1}{5}\frac{G}{c^5}\frac{1}{4}(2\Omega)^6(I_1 - I_2)^2\left\langle \cos^2 2\phi + 2\sin^2 2\phi + \cos^2 2\phi \right\rangle \quad (122)$$

$$= -\frac{32}{5}\frac{G}{c^5}(I_1 - I_2)^2\Omega^6 \quad (123)$$

If we approximate the object with a homogeneous ellipsoid with semiaxes a, b, and c, then

$$I_1 = \frac{1}{5}M(b^2+c^2), \quad I_2 = \frac{1}{5}M(a^2+c^2), \quad I_3 = \frac{1}{5}M(a^2+b^2) \tag{124}$$

and assume a small asymmetry (i.e. $a \approx b$) then we can get

$$\frac{dE}{dt} \approx -\frac{32}{5}\frac{G}{c^5} I_3^2 \epsilon^2 \Omega^6 \tag{125}$$

where the ellipticity ϵ is defined by

$$\epsilon \equiv 2\left(\frac{a-b}{a+b}\right) \tag{126}$$

and

$$h = \frac{16\pi^2 G}{c^4}\frac{\Omega^2}{r}\epsilon I_3 \tag{127}$$

$$= 4\times 10^{-25}\left(\frac{\epsilon}{10^{-6}}\right)\left(\frac{I_3}{10^{45} g\ cm^3}\right)\left(\frac{\Omega}{100Hz}\right)^2\left(\frac{100pc}{r}\right) \tag{128}$$

Remember that $I_3 \approx (2/5)Ma^2 \approx 10^{45}$g cm^3 for $M = 1.4M_\odot$ and $a \approx 10$km.

GW Sources: Slowdown of pulsars

The energy emitted in GWs will be substracted by the rotation energy of the star, ie. the rotational energy will decreases with a rate:

$$\frac{dE_{\rm rot}}{dt} \approx -\frac{32}{5}\frac{G}{c^5} I_3^2 \epsilon^2 \Omega^6 \tag{129}$$

Since the rotation is around the principal axis x_3 the rotational energy will be

$$E_{\rm rot} = \frac{1}{2} I_3 \Omega_{\rm rot}^2 \quad \rightarrow \quad \frac{dE_{\rm rot}}{dt} = I_3 \Omega_{\rm rot} \dot{\Omega}_{\rm rot} \tag{130}$$

and the rotational frequency of the star should decrease as

$$\dot{\Omega}_{\rm rot} = -\frac{32G}{5c^5} \epsilon^2 I_3 \Omega_{\rm rot}^5 \tag{131}$$

Thus, if the slowdown of the pulsar is only due to GW emission we can estimate the deformation and the rotational frequency of the star should decrease as

$$\epsilon^2 = -\frac{5c^5}{32G}\frac{1}{I_3}\frac{\dot{\Omega}_{\rm rot}}{\Omega_{\rm rot}^5} \tag{132}$$

GW Sources: Slowdown of pulsars

The amplitude of the emitted GWs will be:

$$h = 4\pi^2 \left(\frac{5G}{2c^3}\right)^{1/2} \frac{1}{r\sqrt{I_3}} \left(\frac{\dot{\Omega}_{\rm rot}}{\Omega_{\rm rot}}\right)^{1/2} \tag{133}$$

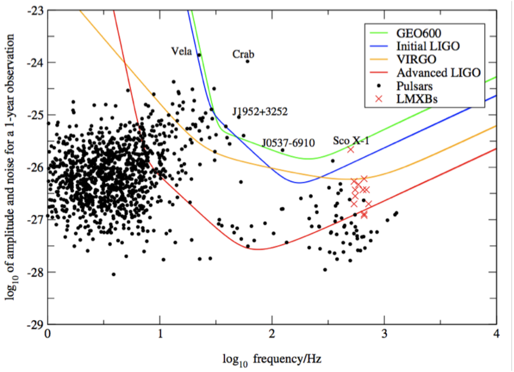


Figure: The strength of the signal of the emitted GWs for the known pulsars (assuming that the slowdown is only due to GE emission).

GW Sources: Slowdown of pulsars

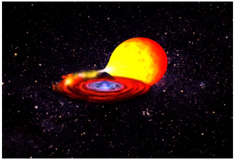
- **Conventional NS crustal shear mountains** : $\epsilon \le 10^{-7} - 10^{-6}$
- **Supefluid vortices** : Magnus-strain deforming crust : $\epsilon \le 5 \times 10^{-7}$
- **Exotic EoS** : strange-quark solid cores
  - Solid quark matter $\epsilon \le 10^{-4}$
  - Quark-baryon mixture of meson condensate matter (half of the core will be solid) $\epsilon \le 10^{-5}$
- **Magnetic mountains**:
  - Large toroidal field 10^{15} G **perpendicular** to rotation : $\epsilon \sim 10^{-6}$
  - Accretion along B-lines $\rightarrow$ "bottled" mountains : $\epsilon \le 10^{-6} - 10^{-5}$

CONCLUDING:

- **Normal nuclear crusts** can only produce ellipticity $\epsilon <$ few $\times 10^{-7}$
- **High ellipticity** measurement means **exotic state of matter**
- **Low ellipticity** is **inconclusive** : strain, buried B-field ...

GW Sources: Slowdown of pulsars

- **Low-mass x-ray binaries (LMXB) are best bet**
  - Rapidly accreting (up to Eddington limit)
  - Rapidly spinning (up to 700Hz) . . . but why not faster?
  - Spin mystery could be nicely solved by GW
- **Emission mechanisms:**
  - Elastic mountains
  - Magnetic mountains
  - r & f-mode oscillations



LIGO searches 2005-2007

- **S2 analysis** : 28 pulsars (all the ones above 50 Hz for which search parameters are "exactly" known)
- **S5 analysis** : 78 pulsars (32 isolated, 41 in binary - 29 in GCs) and $\epsilon \leq 4 \times 10^{-7}$, $h \leq 2 \times 10^{-25}$

GW Detectors : Resonant I

Suppose that a GW propagating along the z-axis with $(+)$ polarization impinges on an idealized detector, two masses joined by a spring along the x-axis

The tidal force induced on the detector is given by equation (64), and the masses will move according to the following equation of motion:

$$\ddot{\xi} + \dot{\xi}/\tau + \omega_0^2 \xi = -\frac{1}{2}\omega^2 L h_+ e^{i\omega t}, \tag{134}$$

where ω_0 is the natural vibration frequency of our detector, τ is the damping time of the oscillator due to frictional forces, L is the separation between the two masses and ξ is the relative change in the distance of the two masses. The GW plays the role of the driving force, and the solution to the above equation is

$$\xi = \frac{\frac{1}{2}\omega^2 L h_+ e^{i\omega t}}{\omega_0^2 - \omega^2 + i\omega/\tau} \tag{135}$$

If the frequency ω of the impinging wave is near the natural frequency ω_0 of the oscillator the detector is excited into large-amplitude motions and it rings like a bell. Actually, in the case of $\omega = \omega_0$, we get the maximum amplitude

$$\xi_{\rm max} = \omega_0 \tau L h_+ / 2. \tag{136}$$

Since the size of our detector L and the amplitude of the gravitational waves h_+ are fixed, large-amplitude motions can be achieved only by increasing the **quality factor** $Q = \omega_0 \tau$ of the detector.
In practice, the frequency of the detector is fixed by its size and the only improvement we can get is by choosing the type of material so that long relaxation times are achieved.

The *cross section* is a measure of the interception ability of a detector. For resonance, the average cross section of our test detector, assuming any possible direction of the wave, is (why?)

$$\sigma = \frac{32\pi}{15} \frac{G}{c^3} \omega_0 Q M L^2. \tag{137}$$

This formula is general; it applies even if we replace our toy detector with a massive metal cylinder.

Weber's first detector. That detector had the following characteristics: Mass M=1410 kg, length L=1.5 m, diameter 66 cm, resonant frequency ω_0=1660Hz, and quality factor $Q = \omega_0 \tau = 2 \times 10^5$. For these values the calculated cross section is roughly $3 \times 10^{-19}\mathrm{cm}^2$.

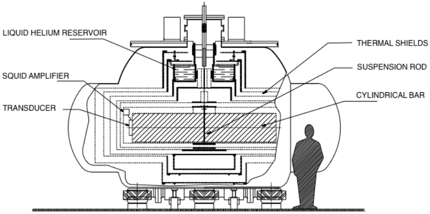


Figure: A graph of NAUTILUS in Frascati near Rome. Nautilus is probably the most sensitive resonant detector available.

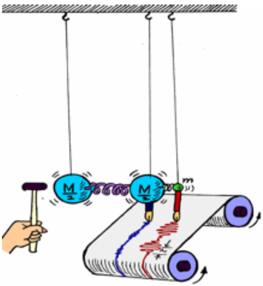
The **thermal noise** is the only factor limiting our ability to detect gravitational waves. Thus, in order to detect a signal, the energy deposited by the GW every τ seconds should be larger than the energy kT due to thermal fluctuations. This leads to a formula for the minimum detectable energy flux of gravitational waves, which, following equation (71), leads into a minimum detectable strain amplitude

$$h_{\min} \geq \frac{1}{\omega_0 L Q}\sqrt{\frac{15kT}{M}} \qquad (138)$$

For Weber's detector, at room temperature this yields a minimum detectable strain of the order of 10^{-20}.
In reality, modern resonant bar detectors are consisting of a solid metallic cylinder suspended *in vacuo* by a cable that is wrapped under its center of gravity. The whole system is cooled down to temperatures of a few K or even mK. To monitor the vibrations of the bar, piezoelectric transducers are attached to the bar. The transducers convert the bar's mechanical energy into electrical energy. The signal is amplified by an ultra-low-frequency amplifier, by using a device called a SQUID (Super-conducting QUantum Interference Device) before it becomes available for data analysis.
The above description of the resonant bar detectors shows that, in order to achieve high sensitivity, one has to:

1. *Create more massive antennas*.
2. *Obtain higher quality factor Q*. Modern antennas generally use aluminum alloy 5056 ($Q \sim 4 \times 10^7$).
3. *Lower the temperature of the antenna as much as possible*. The resonant bar detectors are probably the coolest places in the Universe. Typical cooling temperatures for the most advanced antennae are below the temperature of liquid helium.
4. *Achieve strong coupling between the antenna and the electronics and low electrical noise*.

GW Detectors : Laser Interferometers Sensitivity



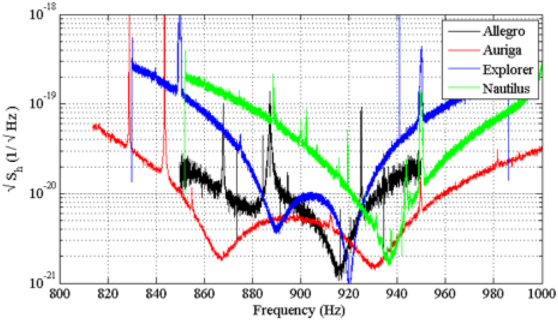


Figure: Present sensitivities of bar detectors.

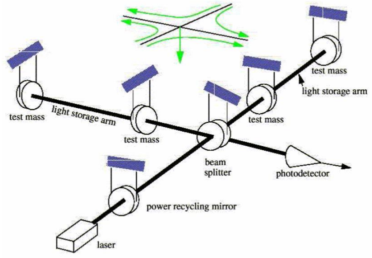
They have achieved sensitivities of a few times 10^{-21}, but still there has been no clear evidence of GW detection. They will have a good chance of detecting a GW signal from a supernova explosion in our galaxy (1-3 events per century). The most sensitive cryogenic bar detectors in operation are:

- ALLEGRO (Baton Rouge, USA) Mass 2296 Kg (Aluminium 5056), length 3 m, bar temperature 4.2 K, mode frequency 896 Hz.
- AURIGA (Legrano, Italy) Mass 2230 Kg (Aluminium 5056), length 2.9 m, bar temperature 0.2 K, mode frequency 913 Hz.
- EXPLORER (CERN, Switzerland) Mass 2270 Kg (Aluminium 5056), length 3 m, bar temperature 2.6K, mode frequency 906Hz.
- NAUTILUS (Frascati, Italy) Mass 2260 Kg (Aluminium 5056), length 3 m, bar temperature 0.1 K, mode frequency 908 Hz.
- NIOBE (Perth, Australia) Mass 1500 Kg (Niobium), length 1.5 m, bar temperature 5K, mode frequency 695Hz.

There are plans for construction of *massive spherical resonant detectors*, the advantages of which will be their high mass, their broader sensitivity (up to 100-200 Hz) and their omnidirectional sensitivity. A prototype spherical detectors are already in operation in Leiden, Italy and Brazil (1 m diameter and mode frequency $\sim$3.2 kHz).

GW Detectors : Laser Interferometers I

A laser interferometer is an alternative GW detector that offers the possibility of very high sensitivities over a broad frequency band.



- Mirrors are attached to M_1 and M_2 and the mirror attached on mass M_0 splits the light (beam splitter) into two perpendicular directions.
- The light is reflected on the two corner mirrors and returns back to the beam splitter.
- The splitter now half-transmits and half-reflects each one of the beams.
- One part of each beam goes back to the laser, while the other parts are combined to reach the photodetector where the fringe pattern is monitored.

GW Detectors : Laser Interferometers II

- Let us consider an impinging GW with amplitude h and $(+)$ polarization, propagating perpendicular to the plane of the detector [9].
- Such a wave will generate a change of $\Delta L \sim hL/2$ in the arm length along the x-direction and an opposite change in the arm length along the y-direction.
- The total difference in length between the two arms will be

$$\frac{\Delta L}{L} \sim h. \qquad (139)$$

- For a GW with amplitude $h \sim 10^{-21}$ and detector arm- length 4 km (such as LIGO), this will induce a change in the arm-length of about $\Delta L \sim 10^{-16}$.
- If the light bounces a few times between the mirrors before it is collected in the photodiode, the effective arm length of the detector is increased considerably, and the measured variations of the arm lengths will be increased accordingly. This is a quite efficient procedure for making the arm length longer.
- The optical cavity that is created between the mirrors of the detector is known as a Fabry-Perot cavity and is used in modern interferometers.

[9] We will further assume that the frequency is much higher than the resonant frequency of the pendulums and the wavelength is much longer than the arm length of our detector

- The passage of a GW changes the length of the arm relative to the other by an amount ΔL. The phase between emerging light beams is changing by

$$\Delta\phi = \frac{2\Delta L}{\tilde{\lambda}} \tag{140}$$

where λ is the wavelength of the light. [10]

- The amplitude of the light signal will be

$$\mathcal{A} \approx 1 + e^{i\pi + \frac{2\Delta L}{\tilde{\lambda}}} \tag{141}$$

- The intensity will be

$$\mathcal{I} \approx \sin^2\left(\frac{\Delta L}{\tilde{\lambda}}\right) \tag{142}$$

- The number of photons that reach the detectors is proportional to the intensity. If the number of photons supplied is $\mathcal{N}_0$, the number of photons that are detected in the emerging light beam is

$$\mathcal{N}_{out} = \mathcal{N}_0 \sin^2(\Delta\phi/2) = \mathcal{N}_0 \sin^2\left(\frac{\Delta L}{\tilde{\lambda}}\right) \tag{143}$$

This equation permits to calculate ΔL from the measurement of the number $\mathcal{N}_{out}$ of the emerging photons.

[10] To achieve maximum sensitivity, it is better to adjust the interferometers in a way that in the absence of GWs the light beams emerging from the two arms are out of phase (destructive interference).

GW Detectors : Photon shot noise

When a GW produces a change ΔL in the arm-length, the phase difference between the two light beams changes by an amount $\Delta\phi = 2b\Delta L/\tilde{\lambda} \sim 10^{-9} rad$ for detectable GWs. [11]

The precision of the measurements, is restricted by fluctuations in the fringe pattern due to fluctuations in the number of detected photons.

The number of detected photons, $\mathcal{N}_{out}$, is (here $\mathcal{N}_0$ is the no of supplied photons) proportional to the intensity of the laser beam

$$\mathcal{N}_{out} = \mathcal{N}_0 \sin^2(\Delta\phi/2) = \mathcal{N}_0 \sin^2(b\Delta L/\tilde{\lambda}) \tag{144}$$

Inversion of this equation leads to an estimation of the relative change of the arm lengths ΔL by measuring the number of the emerging photons $\mathcal{N}_{out}$.

There are statistical fluctuations in the number of detected photons. The magnitude of the fluctuations is

$$\delta\mathcal{N}_{out} = \mathcal{N}_{out}^{1/2} = \mathcal{N}_0^{1/2} \sin(\Delta\phi/2) = \mathcal{N}_0^{1/2} \sin(b\Delta L/\tilde{\lambda})$$

and implies an uncertainty in the measurement of the arm length $\delta(\Delta L) \sim \frac{\tilde{\lambda}}{2b\sqrt{\mathcal{N}_0}}$.

[11] $\tilde{\lambda} \sim 10^{-8} cm$: the reduced wavelength of the laser light & b: the number of bounces of the light in each arm

GW Detectors : Photon shot noise

Thus, the minimum GW amplitude that we can measure (in time τ) is :

$$h_{\min} = \frac{\delta(\Delta L)}{L} = \frac{\Delta L}{L} \sim \frac{\tilde{\lambda}}{bL\mathcal{N}_0^{1/2}} \sim \frac{1}{bL}\left(\frac{\hbar c\tilde{\lambda}}{\tau I_0}\right)^{1/2}, \tag{145}$$

$$h_{\min}(\mathrm{in}\sqrt{Hz}) \sim \frac{1}{bL}\left(\frac{\hbar c\tilde{\lambda}}{I_0}\right)^{1/2}, \tag{146}$$

I_0: intensity of the laser light ($\sim$200 W)
$\tau(\approx 1/\omega)$: the duration of the measurement.

For GWs with frequency 100 Hz we get $h_{\min} \approx 10^{-22}$
while its power spectral density $S_n(f)$ for frequencies 100-200Hz is of the order of $\approx 10^{-23}\sqrt{\mathrm{Hz}}$. [12]

In laser interferometer the photon shot noise dominates for frequencies above 200 Hz.

[12] To express in conventional units $1/\sqrt{Hz}$, one must divide by the square root of the frequency spread, $\sqrt{\Delta\omega} \approx \omega = 10\sqrt{Hz}$.

GW Detectors : Radiation pressure noise

According to (145), the sensitivity of a detector can be increased by increasing the intensity of the laser.
However, a very powerful laser produces a large radiation pressure on the mirrors.

During b reflections by the mirror a photon deposits a momentum $2b \times 2\pi\hbar/\lambda$.

When $\mathcal{N}$ photons strike the mirror, the fluctuation in their number is $\sqrt{\mathcal{N}}$
Then an uncertainty

$$\delta p = 4\pi b\hbar\sqrt{\mathcal{N}}/\lambda \tag{147}$$

in the measurement of the momentum deposited on the mirrors leads to a proportional uncertainty in the position of the mirrors.

Thus, the minimum detectable strain is limited by

$$h_{\rm min} \sim \frac{\tau}{m}\frac{b}{L}\left(\frac{\tau\hbar I_0}{c\tilde{\lambda}}\right)^{1/2}, \tag{148}$$

where m is the mass of the mirrors.

GW Detectors : Shot vs Radiation pressure noise

As we have seen, the photon shot noise decreases as the laser power increases, while the inverse is true for the noise due to radiation pressure fluctuations.

If we try to minimize these two types of noise with respect to the laser power, we get a minimum detectable strain for the optimal power via the very simple relation (how?)

$$h_{\min} \approx \frac{1}{L}\left(\frac{\tau\hbar}{m}\right)^{1/2} \quad \text{for} \quad I_0 = \frac{mc\lambda}{b^2\tau^2} \tag{149}$$

which for the LIGO detector (where the mass of the mirrors is $\sim$100 kg and the arm length is 4 km), for observation time of 1 ms, gives $h_{\min} \approx 10^{-23}$.

GW Detectors : Quantum limit

An additional source of uncertainty in the measurements is set by Heisenberg's principle, which says that the knowledge of the position and the momentum of a body is restricted from the relation $\Delta x \cdot \Delta p \geq \hbar$.

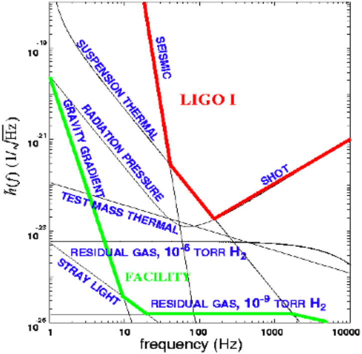
For an observation that lasts some time τ, the smallest measurable displacement of a mirror of mass m is ΔL; assuming that the momentum uncertainty is $\Delta p \approx m \cdot \Delta L/\tau$, we get a minimum detectable strain due to quantum uncertainties

$$h_{\min} = \frac{\Delta L}{L} \sim \frac{1}{L}\left(\frac{\tau\hbar}{m}\right)^{1/2} . \tag{150}$$

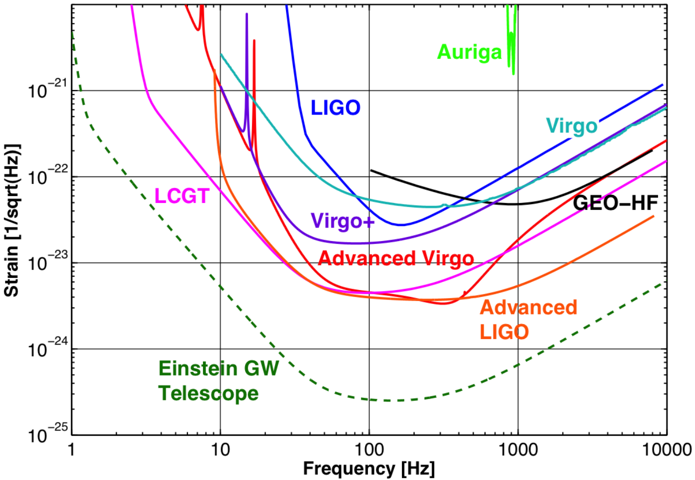
- Surprisingly, this is **identical** to the optimal limit that we calculated earlier for the other two types of noise.
- The standard quantum limit does set a fundamental limit on the sensitivity of beam detectors.
- **An interesting feature of the quantum limit is that it depends only on a single parameter, the mass of the mirrors.**

⋆ *Seismic noise*. At frequencies below 60 Hz, the noise in the interferometers is dominated by seismic noise. The vibrations of the ground couple to the mirrors via the wire suspensions which support them. This effect is strongly suppressed by properly designed suspension systems. Still, seismic noise is very difficult to eliminate at frequencies below 5-10 Hz.

⋆ *Residual gas-phase noise*. The statistical fluctuations of the residual gas density induce a fluctuation of the refraction index and consequently of the monitored phase shift. For this reason the laser beams are enclosed in pipes over their entire length. Inside the pipes a high vacuum of the order of 10^{-9} Torr guarantees elimination of this type of noise.

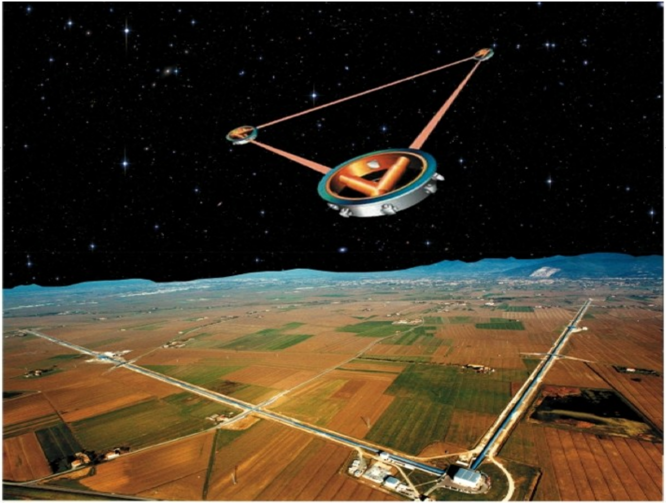


GW: Detectors - Sensitivities

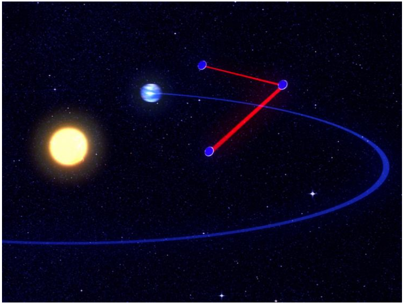


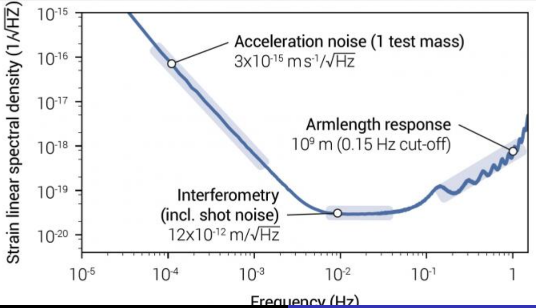
GW Detectors : Ligo Detectors

GW Detectors : Virgo



GW Detectors : eLISA Space Detector





GW Detectors : Laser Interferometers Sensitivity

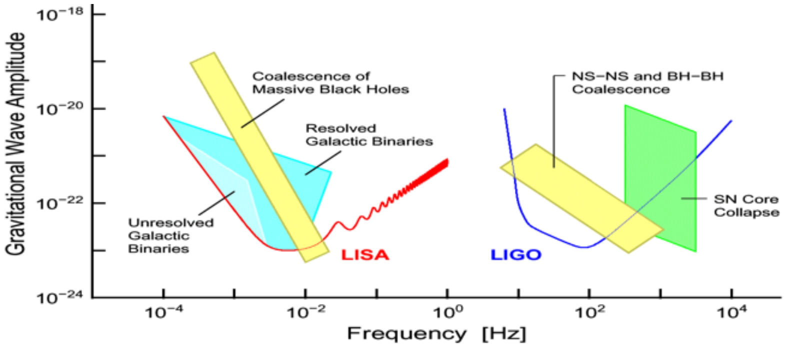
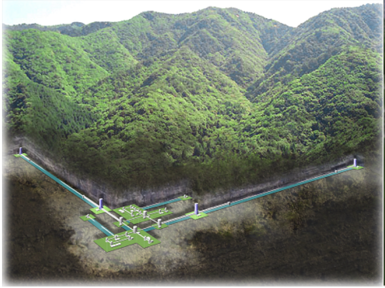
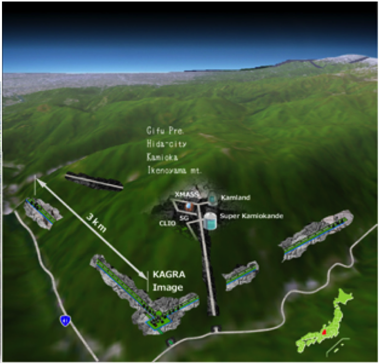


Figure: Sensitivities of laser intereferometers.(ground and space)

GW: Future Detectors (KAGRA)





- KAGRA consists of a modified Michelson interferometer with two 3-km long arms, is located in the ground under Kamioka mine.
- The mirrors are cooled down to cryogenic temperature of -250 Celsius degree (20 Kelvin). Sapphire is chosen for the material of the mirror.
- The goal sensitivity of KAGRA corresponds to observing the moment of coalescence of a binary NS beyond 200 Mpc, or detecting several GW events a year.

GW: Future Detectors (Einstein Telescope)