Spacetime foam and modified dispersion relations

Fabrizio Sorba

Institute for Theoretical Physics
Karlsruhe Institute of Technology

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Objective

- Study how a Lorentz-invariant model of spacetime foam modify the propagation of particles

Spacetime foam

- Quantum mechanics $\Longrightarrow$ quantum fluctuations

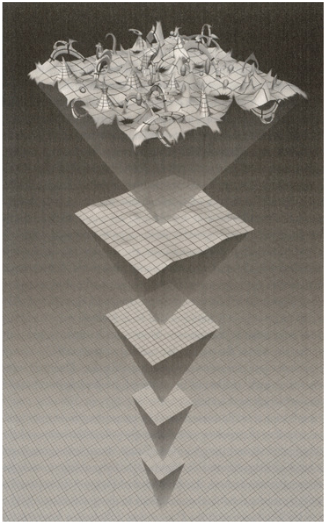
$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

- General relativity:

$$Energy \Longleftrightarrow Geometry$$

- Quantum gravity $\overset{(?)}{\Longrightarrow}$ spacetime foam:

  quantum fluctuations of the geometry and topology of spacetime



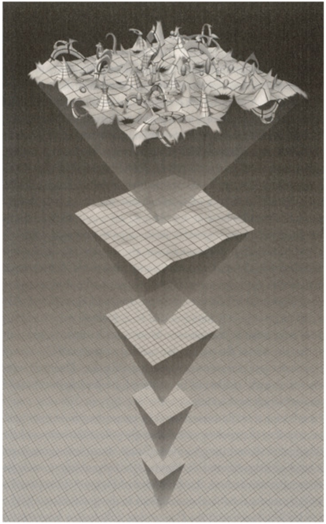
Spacetime foam

- Scale of the fluctuations is expected to be the order of the Planck lenght

$$\delta\ell \sim 10^{-35} m$$

- We take into account only topological fluctuations (defects, wormholes,...)
- We consider defects to be point-like

$$\lambda_{photon} \ll \delta\ell$$



CPT Anomaly[1]

- Chiral gauge theory e.g. $SO(10)$ $(SO(10) \to SU(3) \times SU(2) \times U(1))$
- A single static topological defect (linear defect, wormhole) $\to$ CPT anomaly
- In the $U(1)$ subgroup of electromagnetism the anomalous term turns out to be

$$S_{CPT} = \frac{1}{32\pi} \int d^4x \, f_M(x, A_\mu) \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{1}$$

[1] F. R. Klinkhamer and C. Rupp, "Space-time foam, CPT anomaly, and photon propagation," Phys. Rev. D **70** (2004) 045020 [hep-th/0312032].

CPT Anomaly

- For a large number of defect no exact calculation is practicable
- We assume the abelian anomalies to add up incoherently into a background field:

$$g(x) = \lambda \sum_n \epsilon_n \, h(x - x_n)$$

- $\epsilon_n = \pm 1 \implies \langle g(x) \rangle \sim 0$,
  $h(x - x_n)$: anomalous contribution from the defect at x_n
- The action for the photon field is

$$S_{A_\mu} = -\frac{1}{4} \int d^4x \left\{ F_{\mu\nu} F^{\mu\nu} + g(x) \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} \right\} \quad (2)$$

Sprinkling

- Sprinkling is described as a Poisson process:
  - Divide spacetime into small boxes of volume V
  - Place a sprinkled point into each box with probability $P = \rho V$
  - The limit $V \to 0$ corresponds to the Poisson process
- The result is a Poisson distribution of points into spacetime:

$$P_n(V) = \frac{(\rho V)^n \, e^{-\rho V}}{n!}$$

- $P_n(V)$ is the probability to find n points into the 4-volume V
- The mean value is

$$\langle n(V) \rangle = \sum_n n \, P_n(V) = \rho \, V$$

$\Longrightarrow$ ρ represents the density of sprinkled points

Sprinkling

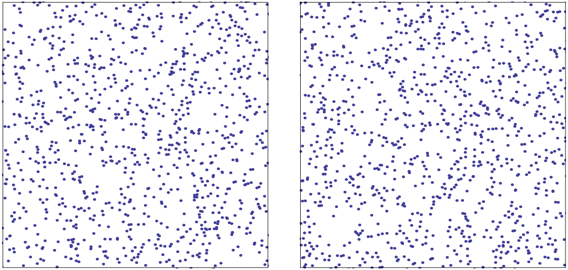
- The Poisson process depends only on the spacetime volume V $\Longrightarrow$ it is Lorentz invariant
- It has been proved that even the individual realizations of the process are Lorentz invariant[2]
- Lorentz invariance here has the following meaning:

  "The discrete set of sprinkled points must not, in and of itself, serve to pick out a preferred reference frame"

[2] L. Bombelli, J. Henson and R. D. Sorkin, "Discreteness without symmetry breaking: A Theorem," Mod. Phys. Lett. A **24** (2009) 2579 [gr-qc/0605006].

Sprinkling

- Example of sprinkling in $2d$ Minkowski spacetime:



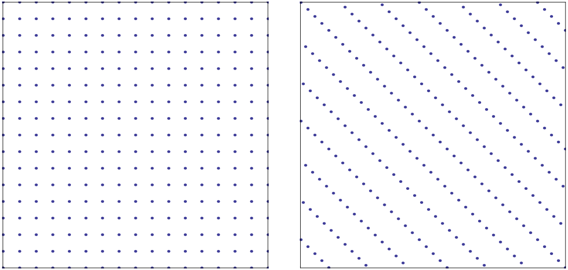
(a) $\beta = 0$ (b) $\beta = 0.7$

Figure: A sprinkling of points as it looks in two different inertial frames

The mean density ρ is the same in both frames

Sprinkling

- A not-invariant distribution:



(a) $\beta = 0$ (b) $\beta = 0.7$

Figure: A regular distribution of points as it looks in two different inertial frames

ρ is uniform in the first case but not in the second

Model

- Our model is based on the effective action:

$$S = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m_0^2}{2} \left(\partial_\mu \phi \partial^\mu \phi - m_1^2 \phi^2 \right) + \right. \\ \left. + \phi \sum_{n=1}^{\infty} \epsilon_n \delta^4(x - x_n) - \frac{\lambda}{4} \phi \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} \right\} \tag{3}$$

- the topological point-like defects are represented by delta functions
- the interaction between defects and photons is mediated by a scalar field

Model

- Solving the scalar field equation (for $\lambda \ll 1$) we obtain the photon action

$$S = -\frac{1}{4}\int d^4x \left\{ F_{\mu\nu}F^{\mu\nu} + g(x)\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} \right\}$$

- in the anomalous term the factor $g(x)$ is

$$g(x) = \int \frac{d^4k}{(2\pi)^4} \left(h(k) \sum_n \epsilon_n e^{ikx_n} \right) e^{-ikx}$$

- where

$$h(k) = \lambda \Delta_\phi(k) = \frac{-\lambda}{m_0^2(k^2 - m_1^2 + i\epsilon)}$$

Model

- The distribution of defects enters in the product $g(q)g(p)$:

$$g(q)g(p) = \lambda^2 \, h(q)h(p) \left(\sum_n e^{i(q+p)x_n} + \sum_{n \neq m} \epsilon_n \epsilon_m e^{iqx_n} e^{ipx_m} \right)$$

- the sum over $n \neq m$ averages to zero, while

$$\sum_n e^{i(q+p)x_n} \simeq \rho \int d^4x \, e^{i(q+p)x} = (2\pi)^4 \rho \, \delta^4(q+p)$$

  sprinkling ensures that $dn = \rho d^4x$ independently of the frame

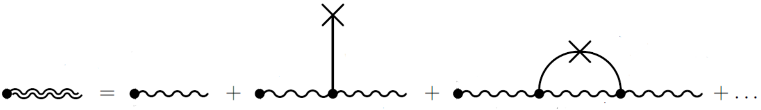
- $$\implies \; g(q)g(p) = (2\pi)^4 \lambda^2 \rho \, h(q)^2 \, \delta^4(q+p) \tag{4}$$

Calculations

- Photon propagator, perturbative expansion:

$$\langle\Omega|A_\mu(a)A_\nu(b)|\Omega\rangle = \langle 0|A_\mu(a)A_\nu(b)|0\rangle - i\int d^4x\,\langle 0|A_\mu(a)A_\nu(b)\mathcal{H}_{int}(x)|0\rangle - \\ - \int d^4x\,d^4y\,\langle 0|A_\mu(a)A_\nu(b)\mathcal{H}_{int}(x)\mathcal{H}_{int}(y)|0\rangle + \dots \tag{5}$$

- $\mathcal{H}_{int}(x) = -\mathcal{L}_{int}(x) = \frac{1}{2}g(x)\varepsilon^{\alpha\beta\rho\sigma}\partial_\alpha A_\beta(x)\partial_\rho A_\sigma(x)$



$$\langle\Omega|A_\mu(a)A_\nu(b)|\Omega\rangle = O(0) + O(\lambda) + O(\lambda^2) + \dots = \\ = D_{\mu\nu}(a-b) + B_{\mu\nu}(a-b) + C_{\mu\nu}(a-b) + \dots \tag{6}$$

Calculations

- $O(\lambda^0)$:

$$D_{\mu\nu}(a-b) = g_{\mu\nu}\Delta_F(k) \qquad \Delta_F(k) = \frac{-i}{k^2 + i\epsilon}$$

- $O(\lambda^1)$:

$$B_{\mu\nu}(a-b) = 0 \qquad (\langle g(x)\rangle = 0)$$

- $O(\lambda^2)$:

$$C_{\mu\nu}(a-b) = \lambda^2 \int \frac{d^4k}{(2\pi)^4} \Delta_F(k)\Pi_{\mu\nu}(k)\, e^{-ik(a-b)}$$

$$\Pi_{\mu\nu}(k) = 3!\rho\, \delta^{\alpha}_{[\gamma}\,\delta^{\rho}_{\eta}\delta^{\beta}_{\nu]} g_{\mu\beta} \int \frac{d^4q}{(2\pi)^4}\, k_\alpha k^\eta (k_\rho - q_\rho)(k^\gamma - q^\gamma)\Delta^2_\phi(q)\Delta_F(k-q)$$

Calculations ($\Pi_{\mu\nu}(k)$)

- We apply the following techniques to manipulate $\Pi_{\mu\nu}(k)$:
  - Passarino-Veltman reduction:

$$\Pi_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi(k)$$

$$\Pi(k) = -\frac{4\rho}{m_0^4} \frac{1}{k^2 + i\epsilon} \int \frac{d^4 q}{(2\pi)^4} \frac{q^2 k^2 - (q \cdot k)^2}{(q^2 - m_1^2 + i\epsilon)^2} \frac{1}{(k-q)^2 + i\epsilon}$$

  - Dimensional regularization:

$$\Pi_{reg}(k) = -\frac{3\rho}{16\pi^2 m_0^4} \left\{ \frac{1}{\hat{\varepsilon}} + \frac{4}{3} - \frac{m_1^2}{k^2} - \log \frac{m_1^2}{\mu^2} - \left(1 - \frac{m_1^2}{k^2} \right)^2 \log \left(1 - \frac{k^2}{m_1^2} - i\epsilon \right) \right\}$$

  - Renormalization ($\Pi_{ren}(k) = \Pi_{reg}(k) - \Pi_{reg}(m_1)$):

$$\boxed{\Pi_{ren}(k) = \frac{3\rho}{16\pi^2 m_0^4} \left\{ \frac{m_1^2}{k^2} + \left(1 - \frac{m_1^2}{k^2} \right)^2 \log \left(1 - \frac{k^2}{m_1^2} - i\epsilon \right) - 1 \right\}}$$

Modified dispersion relations

- To the 2nd order the resummed photon propagator is

$$\overline{D_{\mu\nu}}(k) = g_{\mu\nu}\Delta_F(k)\left(1 - \lambda_{ren}\Pi_{ren}(k) + \lambda_{ren}^2\Pi_{ren}(k)^2 + ...\right) \Rightarrow$$

$$\Rightarrow \overline{D_{\mu\nu}}(k) = \frac{-ig_{\mu\nu}}{k^2(1+\lambda_{ren}^2\Pi_{ren}(k))}$$

- The dispersion relations of the possible wave modes correspond to the poles of the propagator
- The dispersion equation is

$$k^2(1+\lambda_{ren}^2\Pi_{ren}(k)) = 0 \tag{7}$$

Modified dispersion relations, λ real

- $\Pi_{ren}(k)$ is regular in $k^2 = 0$

$$\lim_{k^2 \to 0} k^2 \Pi_{ren}(k) = 0$$

- $\Rightarrow$ one solution is the standard dispersion relation
- $k^2(1 + \lambda^2_{ren}\Pi_{ren}(k)) = 0$ has no other physical solutions
- The only possible dispersion relation is the conventional one,

$$k^2 = 0$$

The foam background does not introduce any modification

Modified dispersion relations, λ imaginary

- $\lambda \longrightarrow i\lambda \quad \Rightarrow \quad \mathcal{H}_{int}(x) \longrightarrow \mathcal{H}'_{int}(x) = i\mathcal{H}_{int}(x)$

$$\mathcal{H}'_{int}(x) = \frac{i}{2} g(x) \varepsilon^{\alpha\beta\rho\sigma} \partial_\alpha A_\beta(x) \partial_\rho A_\sigma(x)$$

- The new Hamiltonian is not more Hermitian, but it is still $\mathcal{PT}$ symmetric ($\mathcal{PT}\,\mathcal{H} = \mathcal{H}$)
- It has been shown[3] that non-Hermitian but $\mathcal{PT}$-symmetric Hamiltonians can coherently describe physical systems
- The dispersion equation is now

$$k^2(1 - \lambda_{ren}^2 \Pi_{ren}(k)) = 0$$

[3] C. M. Bender, S. Boettcher and P. Meisinger, "PT symmetric quantum mechanics," J. Math. Phys. **40** (1999) 2201 [quant-ph/9809072].

Modified dispersion relations, λ imaginary

- In this case a second physically acceptable solution appears:

$$k^2 = \alpha(\gamma) m_1^2 \tag{8}$$

- where $\gamma \propto \lambda^2 \rho$, $\alpha \in [0, 1]$
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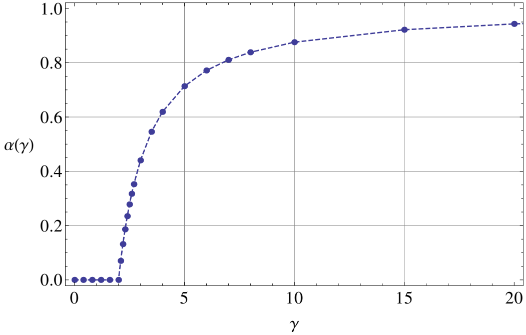

Figure: Several data points for the behaviour of $\alpha(\gamma)$

Modified dispersion relations, percolative foam

- Does $\alpha(\gamma)$ describe a phase transition?

$$\begin{cases} k^2 = 0 & \gamma < \gamma_c \\ k^2 = \alpha(\gamma) m_1^2 & \gamma > \gamma_c \end{cases} \tag{9}$$

- $\gamma \propto \rho$ suggests a relation with percolation phase transition
- Percolation:
  - Percolation theory studies the formation and properties of clusters of objects randomly distributed in space.
  - It exhibits a phase transition:
  - For densities smaller than a critical value ρ_c there are only clusters of finite size
  - For densities larger than ρ_c clusters of infinite size also appear

Modified dispersion relations, percolative foam

- If there is a phase transition then $\alpha(\gamma)$ is the order parameter

$$\Longrightarrow \alpha(\gamma) \propto (1 - \gamma_c/\gamma)^{\beta} \tag{10}$$

- The critical exponent in $4D$ percolation is $\beta = 0.64$
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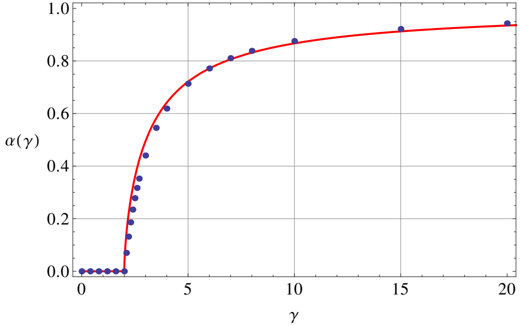

Figure: $\alpha(\gamma)$ interpolated by Eq. (10) with $\beta = 0.64$

Summary

- We studied the propagation of photons through a Lorentz invariant foam of topological point-like defects
  - The topological anomaly is encoded in the anomalous term $\mathcal{L}_{CPT} = -\frac{1}{4}g(x)\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}$
  - Lorentz invariance is garanteed by the sprinkling process
- We found that:
  - For a real coupling constant the foam of defects does not introduce any modification
  - For an imaginary coupling a photon mass seems to emerge

$$m_{photon} = \alpha(\gamma)^{1/2} m_1$$

  - The behaviour of the photon mass respect to the density of defects suggests a relation with the percolation phase transition, but this issue deserves further studies